USA TST 2021/3

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Twitch Solves ISL

Episode 56

Problem

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(y) - \left(\frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z) \right) \le f\left(\frac{x+z}{2} \right) - \frac{f(x)+f(z)}{2}$$

for all real numbers $x < y < z$.

Video

https://youtu.be/A3mS8QrYH6E

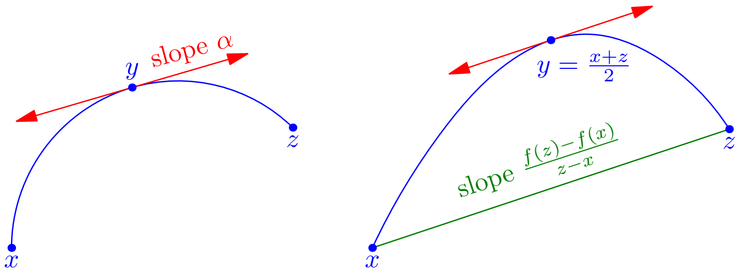
Solution

Answer: all functions of the form $f(y) = ay^2 + by + c$, where a, b, c are constants with $a \le 0$.

If $I = (x, z)$ is an interval, we say that a real number α is a *supergradient* of f at $y \in I$ if we always have

$$f(t) \le f(y) + \alpha(t - y)$$

for every $t \in I$. (This inequality may be familiar as the so-called "tangent line trick". A cartoon of this situation is drawn below for intuition.) We will also say α is a supergradient of f at y, without reference to the interval, if α is a supergradient of *some* open interval containing y.



Claim. The problem condition is equivalent to asserting that $\frac{f(z)-f(x)}{z-x}$ is a supergradient of f at $\frac{x+z}{2}$ for the interval (x, z), for every $x < z$.

Proof. This is just manipulation. □

At this point, we may readily verify that all claimed quadratic functions $f(x) = ax^2 + bx + c$ work — these functions are concave, so the graphs lie below the tangent line at any point. Given $x < z$, the tangent at $\frac{x+z}{2}$ has slope given by the derivative $f'(x) = 2ax + b$, that is

$$f'\left(\frac{x+z}{2}\right) = 2a \cdot \frac{x+z}{2} + b = \frac{f(z) - f(x)}{z - x}$$

as claimed. (Of course, it is also easy to verify the condition directly by elementary means, since it is just a polynomial inequality.)

Now suppose f satisfies the required condition; we prove that it has the above form.

Claim. The function f is concave.

Proof. Choose any $\Delta > \max\{z - y, y - x\}$. Since f has a supergradient α at y over the interval $(y - \Delta, y + \Delta)$, and this interval includes x and z, we have

$$\begin{aligned} \frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z) &\le \frac{z-y}{z-x}(f(y) + \alpha(x-y)) + \frac{y-x}{z-x}(f(y) + \alpha(z-y)) \\ &= f(y). \end{aligned}$$

That is, f is a concave function. Continuity follows from the fact that any concave function on $\mathbb{R}$ is automatically continuous. □

Lemma (see e.g. https://math.stackexchange.com/a/615161 for picture)**.** Any concave function f on $\mathbb{R}$ is continuous.

Proof. Suppose we wish to prove continuity at $p \in \mathbb{R}$. Choose any real numbers a and b with $a < p < b$. For any $0 < \varepsilon < \max(b-p, p-a)$ we always have

$$f(p) + \frac{f(b)-f(p)}{b-p}\varepsilon \le f(p+\varepsilon) \le f(p) + \frac{f(p)-f(a)}{p-a}\varepsilon$$

which implies right continuity; the proof for left continuity is the same. □

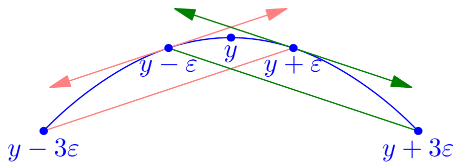
Claim. The function f cannot have more than one supergradient at any given point.

Proof. Fix $y \in \mathbb{R}$. For $t > 0$, let's define the function

$$g(t) = \frac{f(y)-f(y-t)}{t} - \frac{f(y+t)-f(y)}{t}.$$

We contend that $g(\varepsilon) \le \frac{3}{5}g(3\varepsilon)$ for any $\varepsilon > 0$. Indeed by the problem condition,

$$\begin{aligned} f(y) &\le f(y-\varepsilon) + \frac{f(y+\varepsilon)-f(y-3\varepsilon)}{4} \\ f(y) &\le f(y+\varepsilon) - \frac{f(y+3\varepsilon)-f(y-\varepsilon)}{4}. \end{aligned}$$



Summing gives the desired conclusion.

Now suppose that f has two supergradients $\alpha < \alpha'$ at point y. For small enough ε, we should have we have $f(y-\varepsilon) \le f(y) - \alpha'\varepsilon$ and $f(y+\varepsilon) \le f(y) + \alpha\varepsilon$, hence

$$g(\varepsilon) = \frac{f(y)-f(y-\varepsilon)}{\varepsilon} - \frac{f(y+\varepsilon)-f(y)}{\varepsilon} \ge \alpha' - \alpha.$$

This is impossible since $g(\varepsilon)$ may be arbitrarily small. □

Claim. The function f is quadratic on the rational numbers.

Proof. Consider any four-term arithmetic progression $x, x+d, x+2d, x+3d$. Because $(f(x+2d)-f(x+d))/d$ and $(f(x+3d)-f(x))/3d$ are both supergradients of f at the point $x+3d/2$, they must be equal, hence

$$f(x+3d) - 3f(x+2d) + 3f(x+d) - f(x) = 0. \tag{1}$$

If we fix $d = 1/n$, it follows inductively that f agrees with a quadratic function $\widetilde{f}_n$ on the set $\frac{1}{n}\mathbb{Z}$. On the other hand, for any $m \ne n$, we apparently have $\widetilde{f}_n = \widetilde{f}_{mn} = \widetilde{f}_m$, so the quadratic functions on each "layer" are all equal. □

Since f is continuous, it follows f is quadratic, as needed.

Remark (Alternate finish using differentiability due to Michael Ren)**.** In the proof of the main claim (about uniqueness of supergradients), we can actually notice the two terms $\frac{f(y)-f(y-t)}{t}$ and $\frac{f(y+t)-f(y)}{t}$ in the definition of $g(t)$ are both monotonic (by concavity). Since we supplied a proof that $\lim_{t\to 0} g(t) = 0$, we find f is differentiable.

Now, if the derivative at some point exists, it must coincide with all the supergradients; (informally, this is why "tangent line trick" always has the slope as the derivative, and formally, we use the mean value theorem). In other words, we must have

$$f(x+y) - f(x-y) = 2f'(x) \cdot y$$

holds for all real numbers x and y. By choosing $y = 1$ we obtain that $f'(x) = f(x+1) - f(x-1)$ which means f' is also continuous.

Finally differentiating both sides with respect to y gives

$$f'(x+y) - f'(x-y) = 2f'(x)$$

which means f' obeys Jensen's functional equation. Since f' is continuous, this means f' is linear. Thus f is quadratic, as needed.