USA TST 2021/2

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Twitch Solves ISL

Episode 56

Problem

Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.

Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.

Video

https://youtu.be/A3mS8QrYH6E

Solution

For brevity, we let ℓ_i denote line U_iV_i for $i = 1, 2$.

We first give an explicit description of the fixed point K. Let E and F be points on Γ such that $\overline{AE} \parallel \ell_1$ and $\overline{BF} \parallel \ell_2$. The problem conditions imply that E lies between U_1 and A while F lies between U_2 and B. Then we let

$$K = \overline{AF} \cap \overline{BE}.$$

This point exists because $AEFB$ are the vertices of a convex quadrilateral.

Remark (How to identify the fixed point)**.** If we drop the condition that X lies on the arc, then the choice above is motivated by choosing $X \in \{E, F\}$. Essentially, when one chooses $X \to E$, the point C approaches an infinity point. So in this degenerate case, the only points whose power is finite to (XCD) are bounded are those on line BE. The same logic shows that K must lie on line AF. Therefore, if the problem is going to work, the fixed point must be exactly $\overline{AF} \cap \overline{BE}$.

We give two possible approaches for proving the power of K with respect to (XCD) is fixed.

First approach by Vincent Huang We need the following claim:

Claim. Suppose distinct lines AC and BD meet at X. Then for any point K

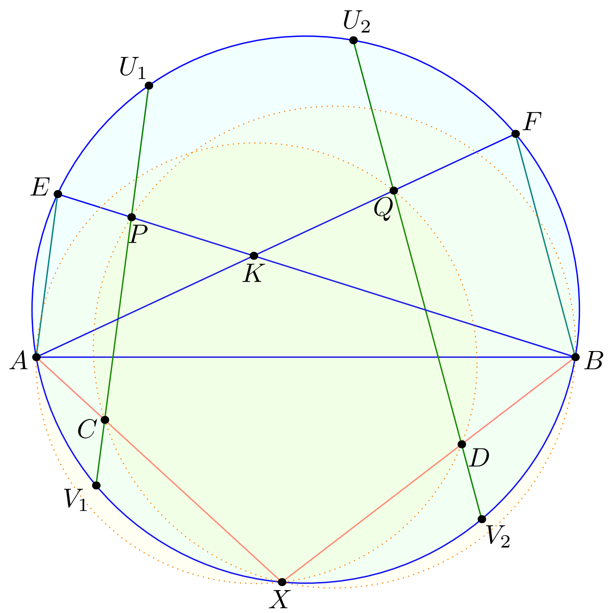
$$\operatorname{pow}(K, XAB) + \operatorname{pow}(K, XCD) = \operatorname{pow}(K, XAD) + \operatorname{pow}(K, XBC).$$

Proof. The difference between the left-hand side and right-hand side is a linear function in K, which vanishes at all of A, B, C, D. □

Construct the points $P = \ell_1 \cap \overline{BE}$ and $Q = \ell_2 \cap \overline{AF}$, which do not depend on X.

Claim. Quadrilaterals $BPCX$ and $AQDX$ are cyclic.

Proof. By Reim's theorem: $\measuredangle CPB = \measuredangle AEB = \measuredangle AXB = \measuredangle CXB$, etc. □



Now, for the particular K we choose, we have

$$\begin{aligned} \operatorname{pow}(K, XCD) &= \operatorname{pow}(K, XAD) + \operatorname{pow}(K, XBC) - \operatorname{pow}(K, XAB) \\ &= KA \cdot KQ + KB \cdot KP - \operatorname{pow}(K, \Gamma). \end{aligned}$$

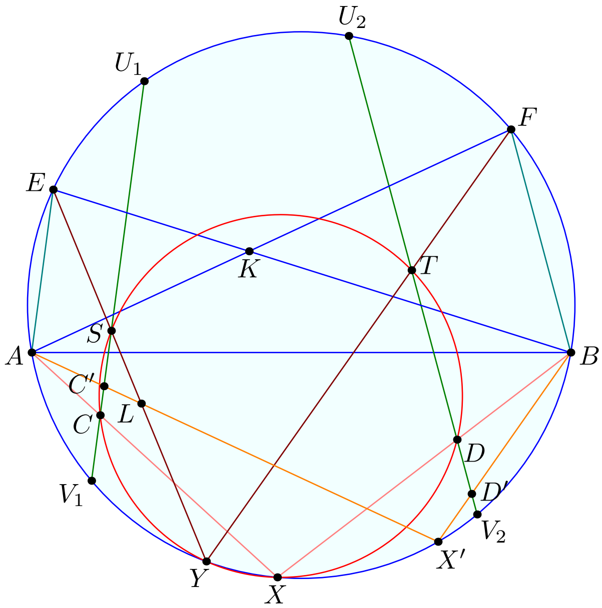
This is fixed, so the proof is completed.

Second approach by authors Let Y be the second intersection of (XCD) with Γ. Let $S = \overline{EY} \cap \ell_1$ and $T = \overline{FY} \cap \ell_2$.

Claim. Points S and T lies on (XCD) as well.

Proof. By Reim's theorem: $\measuredangle CSY = \measuredangle AEY = \measuredangle AXY = \measuredangle CXY$, etc. □

Now let X' be any other choice of X, and define C' and D' in the obvious way. We are going to show that K lies on the radical axis of (XCD) and $(X'C'D')$.



The main idea is as follows:

Claim. The point $L = \overline{EY} \cap \overline{AX'}$ lies on the radical axis. By symmetry, so does the point $M = \overline{FY} \cap \overline{BX'}$ (not pictured).

Proof. Again by Reim's theorem, $SC'YX'$ is cyclic. Hence we have

$$\operatorname{pow}(L, X'C'D') = LC' \cdot LX' = LS \cdot LY = \operatorname{pow}(L, XCD). \quad □$$

To conclude, note that by Pascal theorem on

$$EYFAX'B$$

it follows K, L, M are collinear, as needed.

Remark. All the conditions about U_1, V_1, U_2, V_2 at the beginning are there to eliminate configuration issues, making the problem less obnoxious to the contestant.

In particular, without the various assumptions, there exist configurations in which the point K is at infinity. In these cases, the center of XCD moves along a fixed line.