USA TST 2021/1

Evan Chen

Twitch Solves ISL

Episode 56

Problem

Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

Video

https://youtu.be/J3cOwLGB2ZY

External Link

https://aops.com/community/p20672573

Solution

The answer is s composite.

Composite construction. Write $s = (w+x)(y+z)$, where w, x, y, z are positive integers. Let $a = wy$, $b = wz$, $c = xy$, $d = xz$. Then

$$abc + abd + acd + bcd = wxyz(w+x)(y+z)$$

so this works.

Prime proof. Choose suitable a, b, c, d. Then

$$(a+b)(a+c)(a+d) = (abc + abd + acd + bcd) + a^2(a+b+c+d) \equiv 0 \pmod{s}.$$

Hence s divides a product of positive integers less than s, so s is composite.

Remark. Here is another proof that s is composite.

Suppose that s is prime. Then the polynomial $(x-a)(x-b)(x-c)(x-d) \in \mathbb{F}_s[x]$ is even, so the roots come in two opposite pairs in $\mathbb{F}_s$. Thus the sum of each pair is at least s, so the sum of all four is at least $2s > s$, contradiction.