USA TST 2021/1

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Twitch Solves ISL

Episode 56

Problem

Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

Video

https://youtu.be/A3mS8QrYH6E

Solution

The answer is s composite.

Composite construction Write $s = (w+x)(y+z)$, where w, x, y, z are positive integers. Let $a = wy$, $b = wz$, $c = xy$, $d = xz$. Then

$$abc + abd + acd + bcd = wxyz(w + x)(y + z)$$

so this works.

Prime proof Choose suitable a, b, c, d. Then

$$(a + b)(a + c)(a + d) = (abc + abd + acd + bcd) + a^2(a + b + c + d) \equiv 0 \pmod{s}.$$

Hence s divides a product of positive integers less than s, so s is composite.

Remark. Here is another proof that s is composite.

Suppose that s is prime. Then the polynomial $(x - a)(x - b)(x - c)(x - d) \in \mathbb{F}_s[x]$ is even, so the roots come in two opposite pairs in $\mathbb{F}_s$. Thus the sum of each pair is at least s, so the sum of all four is at least $2s > s$, contradiction.