Iberoamerican 2003/6

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Twitch Solves ISL

Episode 55

Problem

The sequences (a_n), (b_n) are defined by $a_0 = 1, b_0 = 4$ and for $n \geq 0$

$$a_{n+1} = a_n^{2001} + b_n, \qquad b_{n+1} = b_n^{2001} + a_n$$

Show that 2003 is doesn't divide any terms in either sequence.

Video

https://youtu.be/d8d548yI5FA

External Link

https://aops.com/community/p475714

Solution

We prove this the following sentence by induction:

Claim. For $n \geq 0$, $a_n b_n$ is a nonzero quadratic residue modulo 2003.

Proof. The base case is given. For the inductive step, first working modulo 2003 write

$$\begin{aligned} a_{n+1} &= \tfrac{1}{a_n} + b_n \\ b_{n+1} &= \tfrac{1}{b_n} + a_n \end{aligned}$$

and multiply the two equations together to get

$$a_{n+1}b_{n+1} = a_n b_n \left(\frac{1}{a_n b_n} + 1 \right)^2.$$

This proves $a_{n+1}b_{n+1}$ is a quadratic residue.

But, since 2003 is a 3 mod 4 prime, $a_n b_n \not\equiv -1 \bmod 2003$. Hence it follows $a_{n+1}b_{n+1} \not\equiv 0 \pmod{2003}$ and the proof is complete. $\square$