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Twitch Solves ISL

Episode 54

Problem

Given an integer $n > 1$ and its prime factorization $n = p_1^{\alpha 1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, its *false derivative* is defined by

$$f(n) = \alpha_1 p_1^{\alpha_1 - 1} \alpha_2 p_2^{\alpha_2 - 1} \dots \alpha_k p_k^{\alpha_k - 1}.$$

Prove that there exist infinitely many integers $n > 2$ such that $f(n) = f(n-1) + 1$.

Video

https://youtu.be/IMMkiy8rZ9w

External Link

https://aops.com/community/p5469253

Solution

The idea behind the construction is as follows:

Claim. Let m be an integer and let

$$\begin{aligned} x &= 169 \cdot 78m - 25 \\ y &= 27 \cdot 78m - 4. \end{aligned}$$

If x and y are squarefree, then $27x = 169y + 1$ and

$$f(27x) = f(169y) + 1.$$

Proof. Note that $3 \nmid x$ and $13 \nmid y$. Then $f(27x) = 3 \cdot 3^2 = 27$ and $f(169y) = 2 \cdot 13 = 26$, as needed. $\square$

Therefore, it is sufficient to show that there are infinitely many integers m for which x and y as defined above are squarefree.

Fix a large integer M and consider choices of $m \in \{1, \dots, M\}$. For each prime p, the number of m for which $p^2 \mid x$ or $p^2 \mid y$ is at most $2\left\lceil \frac{M}{p^2} \right\rceil$, and is zero if $p^2 > \max(x, y)$. So, the total number of invalid choices of $m \in \{1, \dots, M\}$ is upper bounded by

$$\sum_{p=2}^{O(\sqrt{M})} 2\left\lceil \frac{M}{p^2} \right\rceil < 2M \cdot \sum_{p \ge 2} \frac{1}{p^2} + O\left(\frac{\sqrt{M}}{\log M}\right) < 0.99M$$

for large enough M. This implies the result.