# IGO 2020/A4 

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## Twitch Solves ISL

Episode 52

## Problem

Convex circumscribed quadrilateral $A B C D$ with its incenter $I$ is given such that its incircle is tangent to $\overline{A D}, \overline{D C}, \overline{C B}$, and $\overline{B A}$ at $K, L, M$, and $N$. Let $E=\overline{A D} \cap \overline{B C}$ and $F=\overline{A B} \cap \overline{C D}$. Let $X=\overline{K M} \cap \overline{A B}$ and $Y=\overline{K M} \cap \overline{C D}$. Let $Z=\overline{L N} \cap \overline{A D}$ and $T=\overline{L N} \cap \overline{B C}$.

Prove that the circumcircle of triangle $\triangle X F Y$ and the circle with diameter $E I$ are tangent if and only if the circumcircle of triangle $\triangle T E Z$ and the circle with diameter $F I$ are tangent.

## Video

https://youtu.be/h1P-tSi3Eb8

## Solution

We are going to prove both conditions are equivalent to $\overline{K M} \perp \overline{L N}$.
We introduce the following notation.

- $\Omega$ denotes the circumcircle of $K L M N$.
- Rays $E N$ and $E L$ meet $\Omega$ again at $N^{\prime}$ and $L^{\prime}$.
- Points $X^{\prime}, Y^{\prime}, F^{\prime}, W$ are the midpoints of $\overline{N N^{\prime}}, \overline{L L^{\prime}}, \overline{N L}, \overline{N^{\prime} L^{\prime}}$. Note that $X^{\prime}$ and $Y^{\prime}$ lie on the circle with diameter $\overline{I E}$.
- Let $G=\overline{L N} \cap \overline{K M}$ and $V=\overline{X^{\prime} Y^{\prime}} \cap \overline{M K}$.

The relevance of $F^{\prime}, X^{\prime}, Y^{\prime}$ is explained as follows:
Claim. The points $F^{\prime}, X^{\prime}, Y^{\prime}$ are the inverses of $F, X, Y$ with respect to $\Omega$.
Proof. For $F$ it's clear. For $X$, the inverses of $\overline{M K}$ and $\overline{N N}$ are the circles with diameter $\overline{I E}$ and $\overline{I N}$.

Hence, we are interested in when $\left(F^{\prime} X^{\prime} Y^{\prime}\right)$ is tangent to $\overline{M K}$, the latter being the inverse of the circle with diameter $\overline{E I}$.


The first critical lemma is the following:
Claim (A generalization of butterfly theorem). Unconditionally, we have that ( $G X^{\prime} Y^{\prime}$ ) is tangent to $\overline{M K}$.

Proof. Suppose $\overline{N N^{\prime}} \cap \overline{M K}=X_{1}$ and $\overline{L L^{\prime}} \cap \overline{S M K}=Y_{1}$. Then by shooting lemma from $E$, the quadrilateral $X^{\prime} X_{1} Y_{1} Y^{\prime}$ is cyclic.

Also, by the dual of Desargues involution theorem (DDIT) on $N^{\prime} N L L^{\prime}$, there is an involution swapping $M \leftrightarrow K, X_{1} \leftrightarrow Y_{1}, G \leftrightarrow G$. Combined with power of a point from $V$, we are able to get

$$
V X_{1} \cdot V Y_{1}=V X^{\prime} \cdot V Y^{\prime}=V M \cdot V K \Longrightarrow V X^{\prime} \cdot V Y^{\prime}=V G^{2}
$$

This proves the desired tangency.
Thus, the tangency takes place exactly when $F^{\prime} \in\left(G X^{\prime} Y^{\prime}\right)$.
To prove the result, we now need:
Claim. Unconditionally, $\overline{G F^{\prime}}$ and $\overline{G W}$ are isogonal.
Proof. Since $\triangle G N^{\prime} N \approx \triangle G L L^{\prime}$, we get $\triangle G N^{\prime} X^{\prime} \approx \triangle G L Y^{\prime}$.
Now, $W X^{\prime} F^{\prime} Y^{\prime}$ is always a parallelogram, because these four points are the midpoints of $N N^{\prime} L^{\prime} L$.

So we focus entirely on $\triangle G X^{\prime} Y^{\prime}$ now, and consider the following four statements:

- $\overline{G W}$ and $\overline{G F^{\prime}}$ are isogonal wrt $\angle G$ (and they are distinct, provided $X^{\prime} \neq Y^{\prime} \Longleftrightarrow$ $X \neq Y$, which I think should be assumed for the problem to make sense).
- $W$ and $F^{\prime}$ are reflections are the midpoint of $\overline{X^{\prime} Y^{\prime}}$.
- $F^{\prime}$ lies on $\left(G X^{\prime} Y^{\prime}\right)$.
- $\overline{G W}$ and $\overline{G F^{\prime}}$ are altitude and diameter respectively.

The first two statements are always known; under these two assumptions, one can then show the third and fourth statements are equivalent.

Finally, the last bullet is equivalent to $\overline{G F^{\prime}} \perp \overline{M K}$, because $\overline{M K}$ is a tangent, and hence equivalent to $\overline{L N} \perp \overline{M K}$.

