IGO 2020/A4

Evan Chen

Twitch Solves ISL

Episode 52

Problem

Convex circumscribed quadrilateral $ABCD$ with its incenter I is given such that its incircle is tangent to $\overline{AD}$, $\overline{DC}$, $\overline{CB}$, and $\overline{BA}$ at K, L, M, and N. Let $E = \overline{AD} \cap \overline{BC}$ and $F = \overline{AB} \cap \overline{CD}$. Let $X = \overline{KM} \cap \overline{AB}$ and $Y = \overline{KM} \cap \overline{CD}$. Let $Z = \overline{LN} \cap \overline{AD}$ and $T = \overline{LN} \cap \overline{BC}$.

Prove that the circumcircle of triangle $\triangle XFY$ and the circle with diameter EI are tangent if and only if the circumcircle of triangle $\triangle TEZ$ and the circle with diameter FI are tangent.

Video

https://youtu.be/hlP-tSi3Eb8

Solution

We are going to prove both conditions are equivalent to $\overline{KM} \perp \overline{LN}$.

We introduce the following notation.

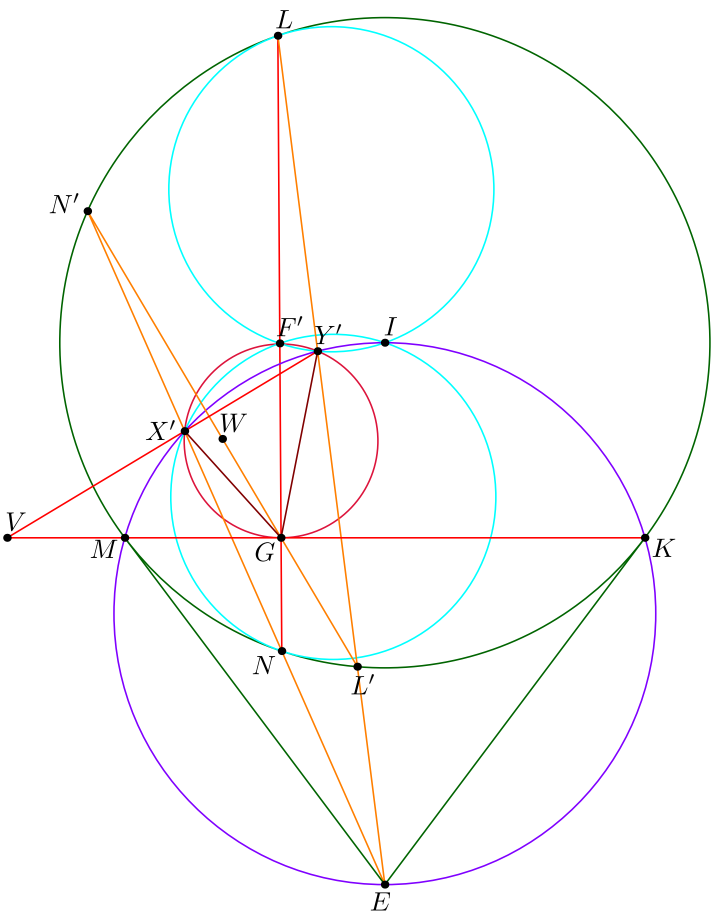
- Ω denotes the circumcircle of $KLMN$.
- Rays EN and EL meet Ω again at N' and L'.
- Points X', Y', F', W are the midpoints of $\overline{NN'}$, $\overline{LL'}$, $\overline{NL}$, $\overline{N'L'}$. Note that X' and Y' lie on the circle with diameter $\overline{IE}$.
- Let $G = \overline{LN} \cap \overline{KM}$ and $V = \overline{X'Y'} \cap \overline{MK}$.

The relevance of F', X', Y' is explained as follows:

Claim. The points F', X', Y' are the inverses of F, X, Y with respect to Ω.

Proof. For F it's clear. For X, the inverses of $\overline{MK}$ and $\overline{NN}$ are the circles with diameter $\overline{IE}$ and $\overline{IN}$. □

Hence, we are interested in when $(F'X'Y')$ is tangent to $\overline{MK}$, the latter being the inverse of the circle with diameter $\overline{EI}$.



The first critical lemma is the following:

Claim (A generalization of butterfly theorem)**.** Unconditionally, we have that $(GX'Y')$ is tangent to $\overline{MK}$.

Proof. Suppose $\overline{NN'} \cap \overline{MK} = X_1$ and $\overline{LL'} \cap \overline{SMK} = Y_1$. Then by shooting lemma from E, the quadrilateral $X'X_1Y_1Y'$ is cyclic.

Also, by the dual of Desargues involution theorem (DDIT) on $N'NLL'$, there is an involution swapping $M \leftrightarrow K$, $X_1 \leftrightarrow Y_1$, $G \leftrightarrow G$. Combined with power of a point from V, we are able to get

$$VX_1 \cdot VY_1 = VX' \cdot VY' = VM \cdot VK \implies VX' \cdot VY' = VG^2.$$

This proves the desired tangency. □

Thus, the tangency takes place exactly when $F' \in (GX'Y')$.

To prove the result, we now need:

Claim. Unconditionally, $\overline{GF'}$ and $\overline{GW}$ are isogonal.

Proof. Since $\triangle GN'N \overset{-}{\sim} \triangle GLL'$, we get $\triangle GN'X' \overset{-}{\sim} \triangle GLY'$. □

Now, $WX'F'Y'$ is always a parallelogram, because these four points are the midpoints of $NN'L'L$.

So we focus entirely on $\triangle GX'Y'$ now, and consider the following four statements:

- $\overline{GW}$ and $\overline{GF'}$ are isogonal wrt $\angle G$ (and they are distinct, provided $X' \neq Y' \iff X \neq Y$, which I think should be assumed for the problem to make sense).
- W and F' are reflections are the midpoint of $\overline{X'Y'}$.
- F' lies on $(GX'Y')$.
- $\overline{GW}$ and $\overline{GF'}$ are altitude and diameter respectively.

The first two statements are always known; under these two assumptions, one can then show the third and fourth statements are equivalent.

Finally, the last bullet is equivalent to $\overline{GF'} \perp \overline{MK}$, because $\overline{MK}$ is a tangent, and hence equivalent to $\overline{LN} \perp \overline{MK}$.