## IGO 2020/A4 Evan Chen

TWITCH SOLVES ISL

Episode 52

## Problem

Convex circumscribed quadrilateral ABCD with its incenter I is given such that its incircle is tangent to  $\overline{AD}$ ,  $\overline{DC}$ ,  $\overline{CB}$ , and  $\overline{BA}$  at K, L, M, and N. Let  $E = \overline{AD} \cap \overline{BC}$  and  $F = \overline{AB} \cap \overline{CD}$ . Let  $X = \overline{KM} \cap \overline{AB}$  and  $Y = \overline{KM} \cap \overline{CD}$ . Let  $Z = \overline{LN} \cap \overline{AD}$  and  $T = \overline{LN} \cap \overline{BC}$ .

Prove that the circumcircle of triangle  $\triangle XFY$  and the circle with diameter EI are tangent if and only if the circumcircle of triangle  $\triangle TEZ$  and the circle with diameter FI are tangent.

## Video

https://youtu.be/hlP-tSi3Eb8

## Solution

We are going to prove both conditions are equivalent to  $\overline{KM} \perp \overline{LN}$ . We introduce the following notation.

- $\Omega$  denotes the circumcircle of *KLMN*.
- Rays EN and EL meet  $\Omega$  again at N' and L'.
- Points X', Y', F', W are the midpoints of  $\overline{NN'}$ ,  $\overline{LL'}$ ,  $\overline{NL}$ ,  $\overline{N'L'}$ . Note that X' and Y' lie on the circle with diameter  $\overline{IE}$ .
- Let  $G = \overline{LN} \cap \overline{KM}$  and  $V = \overline{X'Y'} \cap \overline{MK}$ .

The relevance of F', X', Y' is explained as follows:

**Claim.** The points F', X', Y' are the inverses of F, X, Y with respect to  $\Omega$ .

*Proof.* For F it's clear. For X, the inverses of  $\overline{MK}$  and  $\overline{NN}$  are the circles with diameter  $\overline{IE}$  and  $\overline{IN}$ .

Hence, we are interested in when (F'X'Y') is tangent to  $\overline{MK}$ , the latter being the inverse of the circle with diameter  $\overline{EI}$ .



The first critical lemma is the following:

Claim (A generalization of butterfly theorem). Unconditionally, we have that (GX'Y') is tangent to  $\overline{MK}$ .

*Proof.* Suppose  $\overline{NN'} \cap \overline{MK} = X_1$  and  $\overline{LL'} \cap \overline{SMK} = Y_1$ . Then by shooting lemma from E, the quadrilateral  $X'X_1Y_1Y'$  is cyclic.

Also, by the dual of Desargues involution theorem (DDIT) on N'NLL', there is an involution swapping  $M \leftrightarrow K$ ,  $X_1 \leftrightarrow Y_1$ ,  $G \leftrightarrow G$ . Combined with power of a point from V, we are able to get

$$VX_1 \cdot VY_1 = VX' \cdot VY' = VM \cdot VK \implies VX' \cdot VY' = VG^2.$$

This proves the desired tangency.

Thus, the tangency takes place exactly when  $F' \in (GX'Y')$ . To prove the result, we now need:

Claim. Unconditionally,  $\overline{GF'}$  and  $\overline{GW}$  are isogonal.

*Proof.* Since  $\triangle GN'N \sim \triangle GLL'$ , we get  $\triangle GN'X' \sim \triangle GLY'$ .

Now, WX'F'Y' is always a parallelogram, because these four points are the midpoints of NN'L'L.

So we focus entirely on  $\triangle GX'Y'$  now, and consider the following four statements:

- $\overline{GW}$  and  $\overline{GF'}$  are isogonal wrt  $\angle G$  (and they are distinct, provided  $X' \neq Y' \iff X \neq Y$ , which I think should be assumed for the problem to make sense).
- W and F' are reflections are the midpoint of  $\overline{X'Y'}$ .
- F' lies on (GX'Y').
- $\overline{GW}$  and  $\overline{GF'}$  are altitude and diameter respectively.

The first two statements are always known; under these two assumptions, one can then show the third and fourth statements are equivalent.

Finally, the last bullet is equivalent to  $\overline{GF'} \perp \overline{MK}$ , because  $\overline{MK}$  is a tangent, and hence equivalent to  $\overline{LN} \perp \overline{MK}$ .