AMC 12A 2021/24

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Twitch Solves ISL

Episode 52

Problem

Semicircle Γ has diameter $\overline{AB}$ of length 14. Circle Ω lies tangent to $\overline{AB}$ at a point P and intersects Γ at points Q and R. If $QR = 3\sqrt{3}$ and $\angle QPR = 60^\circ$, find the area of $\triangle PQR$.

Video

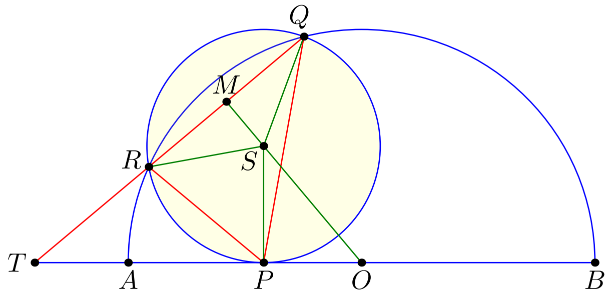
https://youtu.be/0-fh01O_Zao

External Link

https://aops.com/community/p23613274

Solution

Let S be the center of Ω, O the center of Γ, and $T = \overline{QR} \cap \overline{APOB}$. Let $M = \overline{OS} \cap \overline{QR}$ be the midpoint of $\overline{QR}$.



Since $QR = 3\sqrt{3}$ and $\angle QSR = 2\angle QPR = 120^\circ$, it follows that $SP = SQ = SR = 3$. Also, $SM = SQ \sin \angle QSM = 3 \sin 60^\circ = \frac{3}{2}$. Therefore,

$$\begin{aligned} OS = OM - SM &= \sqrt{OR^2 - \left(\frac{1}{2}QR\right)^2} - SM \\ &= \sqrt{7^2 - \frac{27}{4}} - \frac{3}{2} = 5. \end{aligned}$$

So $\triangle OSP$ is a $3-4-5$ triangle.

Now, $\triangle OTM \sim \triangle OSP$, so one can compute

$$\begin{aligned} TO &= \frac{5}{4} \cdot OM = \frac{5}{4} \cdot \frac{13}{2} = \frac{65}{8} \\ TP &= TO - PO = \frac{65}{8} - 4 = \frac{33}{8}. \end{aligned}$$

So the height from P to $\overline{TM}$ equals

$$OM \cdot \frac{TP}{TO} = \frac{13}{2} \cdot \frac{33/8}{65/8} = \frac{33}{10}.$$

Finally, $[PQR] = \frac{1}{2}QR \cdot \frac{33}{10} = \frac{99\sqrt{3}}{20}$.