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TWITCH SOLVES ISL

Episode 51

Problem

A function $f\colon \mathbb{Z} \to \mathbb{Z}$ is given. Suppose that for every integer $n \geq 0$, there are at most $0.001n^2$ pairs of integers (x, y) for which $f(x+y) \neq f(x) + f(y)$ and $\max\{|x|, |y|\} \leq n$. Is it possible that for some integer $n \geq 0$, there are more than n integers a such that $f(a) \neq a \cdot f(1)$ and $|a| \leq n$?

Video

https://youtu.be/qyfqm9u2ulQ

Solution

The answer is no.

Let's say an integer is *bad* if $f(n) \neq n \cdot f(1)$. It is not hard to see that any bad integer must have absolute value at least 30.

Let's say a pair of integers is *bad* if $f(x+y) \neq f(x) + f(y)$.

Claim. Let $n \geq 1$, and suppose there are εn bad integers in $[-n, n]$. Then

$$\varepsilon\,(1 - 2\varepsilon) < 0.001.$$

Proof. If $\varepsilon = 0$ there is nothing to prove.

Otherwise, pick a bad integer b and consider

$$\ell_b = \{(x, y) \mid x, y \in [-n, n], x + y = b\}\,.$$

(We call this a "line" colloquially, thinking of it in the xy-plane.) Note that:

- There are at least n points on ℓ_b.
- For every $(x, y) \in \ell_b$, either the pair itself is bad, or one of x or y is bad. (Otherwise, putting (x, y) in the functional equation would contradict badness of b).
- Therefore, there are at least $n - 2\varepsilon n$ bad pairs on ℓ_b.

Taking the union across all εn lines ℓ_b, we conclude the total number of bad pairs is at least

$$\varepsilon n \cdot (1 - 2\varepsilon)n.$$

This gives the desired result. □

Note that the solution to the above quadratic is $\varepsilon < 0.001002$ and $\varepsilon > 0.498998$ (decimal approximations for convenience).

Viewing ε as a function $\varepsilon(n)$ now, we not that $\varepsilon(n) = 0$ initially for $n = 1, 2, \dots, 30$. There is no way for $\varepsilon(n)$ to "jump" in the sense that $\varepsilon(n) < 0.1$ and $\varepsilon(n+1) > 0.4$, so this ends the proof.