USAMO 1997/6

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Twitch Solves ISL

Episode 50

Problem

Suppose the sequence of nonnegative integers $a_1, a_2, \dots, a_{1997}$ satisfies

$$a_i + a_j \le a_{i+j} \le a_i + a_j + 1$$

for all $i, j \ge 1$ with $i + j \le 1997$. Show that there exists a real number x such that $a_n = \lfloor nx \rfloor$ for all $1 \le n \le 1997$.

Video

https://youtu.be/q4n-74-t1xY

Solution

We are trying to show there exists an $x \in \mathbb{R}$ such that

$$\frac{a_n}{n} \le x < \frac{a_n + 1}{n} \qquad \forall n.$$

This means we need to show

$$\max_i \frac{a_i}{i} < \min_j \frac{a_j + 1}{j}.$$

Replace 1997 by N. We will prove this by induction, but we will need some extra hypotheses on the indices i, j which are used above.

Claim. Suppose that

- Integers $a_1, a_2, \dots, a_N$ satisfy the given conditions.
- Let $i = \operatorname{argmax}_n \frac{a_n}{n}$; if there are ties, pick the smallest i.
- Let $j = \operatorname{argmin}_n \frac{a_n+1}{n}$; if there are ties, pick the smallest j.

Then

$$\frac{a_i}{i} < \frac{a_j + 1}{j}.$$

Moreover, these two fractions are in lowest terms, and are adjacent in the Farey sequence of order $\max(i, j)$.

Proof. By induction on $N \ge 1$ with the base case clear. So suppose we have the induction hypothesis with numbers $a_1, \dots, a_{N-1}$, with i and j as promised.

Now, consider the new number a_N. We have two cases:

- Suppose $i + j > N$. Then, no fraction with denominator N can lie strictly inside the interval; so we may write for some integer b

  $$\frac{b}{N} \le \frac{a_i}{i} < \frac{a_j + 1}{j} \le \frac{b+1}{N}.$$

  For purely algebraic reasons we have

  $$\frac{b - a_i}{N - i} \le \frac{b}{N} \le \frac{a_i}{i} < \frac{a_j + 1}{j} \le \frac{b+1}{N} \le \frac{b - a_j}{N - j}.$$

  Now,

  $$\begin{aligned} a_N &\ge a_i + a_{N-i} \ge a_i + (N - i) \cdot \frac{a_i}{i} \\ &\ge a_i + (b - a_i) = b \\ a_N &\le a_j + a_{N-j} + 1 \le (a_j + 1) + (N - j) \cdot \frac{a_j + 1}{j} \\ &= (a_j + 1) + (b - a_j) = b + 1. \end{aligned}$$

  Thus $a_N \in \{b, b+1\}$. This proves that $\frac{a_N}{N} \le \frac{a_i}{i}$ while $\frac{a_N+1}{N} \ge \frac{a_j+1}{j}$. Moreover, the pair (i, j) does not change, so all inductive hypotheses carry over.

- On the other hand, suppose $i + j = N$. Then we have

$$\frac{a_i}{i} < \frac{a_i + a_j + 1}{N} < \frac{a_j + 1}{j}.$$

Now, we know a_N could be either $a_i + a_j$ or $a_i + a_j + 1$. If it's the former, then (i, j) becomes (i, N). If it's the latter, then (i, j) becomes (N, j). The properties of Farey sequences ensure that the $\frac{a_i+a_j+1}{N}$ is reduced, either way.

□