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Twitch Solves ISL

Episode 50

Problem

Let $p_1, p_2, p_3, \ldots$ be the prime numbers listed in increasing order, and let $0 < x_0 < 1$ be a real number between 0 and 1. For each positive integer k, define

$$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0, \\ \left\{ \dfrac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0, \end{cases}$$

where $\{x\}$ denotes the fractional part of x. Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \ldots$ eventually becomes 0.

Video

https://youtu.be/5jIFaUjnkg0

Solution

The answer is x_0 rational.

If x_0 is irrational, then all x_i are irrational by induction. So the sequence cannot become zero.

If x_0 is rational, then all are. Now one simply observes that the denominators of x_n are strictly decreasing, until we reach $0 = \frac{0}{1}$. This concludes the proof.

Remark. The sequence p_k could have been any sequence of integers.