TSTST 2020/9

Evan Chen

Twitch Solves ISL

Episode 50

Problem

Ten million fireflies are glowing in $\mathbb{R}^3$ at midnight. Some of the fireflies are friends, and friendship is always mutual. Every second, one firefly moves to a new position so that its distance from each one of its friends is the same as it was before moving. This is the only way that the fireflies ever change their positions. No two fireflies may ever occupy the same point.

Initially, no two fireflies, friends or not, are more than a meter away. Following some finite number of seconds, all fireflies find themselves at least ten million meters away from their original positions. Given this information, find the greatest possible number of friendships between the fireflies.

Video

https://youtu.be/jsw3c3yAn7o

Solution

In general, we show that when $n \geq 70$, the answer is $f(n) = \lfloor \frac{n^2}{3} \rfloor$.

Construction: Choose three pairwise parallel lines ℓ_A, ℓ_B, ℓ_C forming an infinite equilateral triangle prism (with side larger than 1). Split the n fireflies among the lines as equally as possible, and say that two fireflies are friends iff they lie on different lines.

To see this works:

1. Reflect ℓ_A and all fireflies on ℓ_A in the plane containing ℓ_B and ℓ_C.
2. Reflect ℓ_B and all fireflies on ℓ_B in the plane containing ℓ_C and ℓ_A.
3. Reflect ℓ_C and all fireflies on ℓ_C in the plane containing ℓ_A and ℓ_B.

$\vdots$

Proof: Consider a valid configuration of fireflies. If there is no 4-clique of friends, then by Turán's theorem, there are at most $f(n)$ pairs of friends.

Let $g(n)$ be the answer, given that there exist four pairwise friends (say a, b, c, d). Note that for a firefly to move, all its friends must be coplanar.

Claim (No coplanar K_4)**.** We can't have four coplanar fireflies which are pairwise friends.

Proof. If we did, none of them could move (unless three are collinear, in which case they can't move). $\square$

Claim (Key claim — tetrahedrons don't share faces often)**.** There are at most 12 fireflies e which are friends with at least three of a, b, c, d.

Proof. First denote by A, B, C, D the locations of fireflies a, b, c, d. These four positions change over time as fireflies move, but the tetrahedron $ABCD$ always has a fixed shape, and we will take this tetrahedron as our reference frame for the remainder of the proof.

WLOG, will assume that e is friends with a, b, c. Then e will always be located at one of two points E_1 and E_2 relative to ABC, such that E_1ABC and E_2ABC are two congruent tetrahedrons with fixed shape. We note that points D, E_1, and E_2 are all different: clearly $D \neq E_1$ and $E_1 \neq E_2$. (If $D = E_2$, then some fireflies won't be able to move.)

Consider the moment where firefly a moves. Its friends must be coplanar at that time, so one of E_1, E_2 lies in plane BCD. Similar reasoning holds for planes ACD and ABD.

So, WLOG E_1 lies on both planes BCD and ACD. Then E_1 lies on line CD, and E_2 lies in plane ABD. This uniquely determines (E_1, E_2) relative to $ABCD$:

- E_1 is the intersection of line CD with the reflection of plane ABD in plane ABC.
- E_2 is the intersection of plane ABD with the reflection of line CD in plane ABC.

Accounting for WLOGs, there are at most 12 possibilities for the set $\{E_1, E_2\}$, and thus at most 12 possibilities for E. (It's not possible for both elements of one pair $\{E_1, E_2\}$ to be occupied, because then they couldn't move.) $\square$

Thus, the number of friendships involving exactly one of a, b, c, d is at most $(n-16)\cdot 2+12\cdot 3=2n+4$, so removing these four fireflies gives

$$g(n)\le 6+(2n+4)+\max\{f(n-4),g(n-4)\}.$$

The rest of the solution is bounding. When $n\ge 24$, we have $(2n+10)+f(n-4)\le f(n)$, so

$$g(n)\le \max\{f(n),(2n+10)+g(n-4)\}\quad \forall n\ge 24.$$

By iterating the above inequality, we get

$$\begin{aligned} g(n)\le \max\Big\{f(n),\ &(2n+10)+(2(n-4)+10) \\ &+\cdots+(2(n-4r)+10)+g(n-4r-4)\Big\}, \end{aligned}$$

where r satisfies $n-4r-4<24\le n-4r$.

Now

$$\begin{aligned} &(2n+10)+(2(n-4)+10)+\cdots+(2(n-4r)+10)+g(n-4r-4) \\ &=(r+1)(2n-4r+10)+g(n-4r-4) \\ &\le\left(\frac{n}{4}-5\right)(n+37)+\binom{24}{2}. \end{aligned}$$

This is less than $f(n)$ for $n\ge 70$, which concludes the solution.

Remark. There are positive integers n such that it is possible to do better than $f(n)$ friendships. For instance, $f(5)=8$, whereas five fireflies a, b, c, d, and e as in the proof of the Lemma (E_1 being the intersection point of line CD with the reflection of plane (ABD) in plane (ABC), E_2 being the intersection point of plane (ABD) with the reflection of line CD in plane (ABC), and tetrahedron $ABCD$ being sufficiently arbitrary that points E_1 and E_2 exist and points D, E_1, and E_2 are pairwise distinct) give a total of nine friendships.

Remark (Author comments)**.** It is natural to approach the problem by looking at the two-dimensional version first. In two dimensions, the following arrangement suggests itself almost immediately: We distribute all fireflies as equally as possible among two parallel lines, and two fireflies are friends if and only if they are on different lines.

Similarly to the three-dimensional version, this attains the greatest possible number of friendships for all sufficiently large n, though not for all n. For instance, at least one friendlier arrangements exists for $n=4$, similarly to the above friendlier arrangement for $n=5$ in three dimensions.

This observation strongly suggests that in three dimensions we should distribute the fireflies as equally as possible among two parallel planes, and that two fireflies should be friends if and only if they are on different planes. It was a great surprise for me to discover that this arrangement does not in fact give the correct answer!

Remark. On the other hand, Ankan Bhattacharya gives the following reasoning as to why the answer should not be that surprising:

> I think the answer $(10^{14}-1)/3$ is quite natural if you realize that $(n/2)^2$ is probably optimal in 2D and $\binom{n}{2}$ is optimal in super high dimensions (i.e. around n). So going from dimension 2 to 3 should increase the answer (and indeed it does).