TSTST 2020/8 Evan Chen

Twitch Solves ISL

Episode 50

Problem

For every positive integer N, let $\sigma(N)$ denote the sum of the positive integer divisors of N. Find all integers $m \ge n \ge 2$ satisfying

$$\frac{\sigma(m) - 1}{m - 1} = \frac{\sigma(n) - 1}{n - 1} = \frac{\sigma(mn) - 1}{mn - 1}.$$

Video

https://youtu.be/NOjZ_DKUgxc

External Link

https://aops.com/community/p20020195

Solution

The answer is that m and n should be powers of the same prime number. These all work because for a prime power we have

$$\frac{\sigma(p^e) - 1}{p^e - 1} = \frac{(1 + p + \dots + p^e) - 1}{p^e - 1} = \frac{p(1 + \dots + p^{e-1})}{p^e - 1} = \frac{p}{p - 1}$$

So we now prove these are the only ones. Let λ be the common value of the three fractions.

Claim. Any solution (m, n) should satisfy d(mn) = d(m) + d(n) - 1.

Proof. The divisors of mn include the divisors of m, plus m times the divisors of n (counting m only once). Let λ be the common value; then this gives

$$\sigma(mn) \ge \sigma(m) + m\sigma(n) - m$$

= $(\lambda m - \lambda + 1) + m(\lambda n - \lambda + 1) - m$
= $\lambda mn - \lambda + 1$

and so equality holds. Thus these are all the divisors of mn, for a count of d(m) + d(n) - 1.

Claim. If d(mn) = d(m) + d(n) - 1 and $\min(m, n) \ge 2$, then m and n are powers of the same prime.

Proof. Let A denote the set of divisors of m and B denote the set of divisors of n. Then $|A \cdot B| = |A| + |B| - 1$ and $\min(|A|, |B|) > 1$, so |A| and |B| are geometric progressions with the same ratio. It follows that m and n are powers of the same prime.

Remark (Nikolai Beluhov). Here is a completion not relying on $|A \cdot B| = |A| + |B| - 1$. By the above arguments, we see that every divisor of mn is either a divisor of n, or n times a divisor of m.

Now suppose that some prime $p \mid m$ but $p \nmid n$. Then $p \mid mn$ but p does not appear in the above classification, a contradiction. By symmetry, it follows that m and n have the same prime divisors.

Now suppose we have different primes $p \mid m$ and $q \mid n$. Write $\nu_p(m) = \alpha$ and $\nu_p(n) = \beta$. Then $p^{\alpha+\beta} \mid mn$, but it does not appear in the above characterization, a contradiction. Thus, m and n are powers of the same prime.

Remark (Comments on the function in the problem). Let $f(n) = \frac{\sigma(n)-1}{n-1}$. Then f is not really injective even outside the above solution; for example, we have $f(6 \cdot 11^k) = \frac{11}{5}$ for all k, plus sporadic equivalences like f(14) = f(404), as pointed out by one reviewer during test-solving. This means that both relations should be used at once, not independently.

Remark (Authorship remarks). Ankan gave the following story for how he came up with the problem while thinking about so-called *almost perfect* numbers.

I was in some boring talk when I recalled a conjecture that if $\sigma(n) = 2n - 1$, then n is a power of 2. For some reason (divine intervention, maybe) I had the double idea of (1) seeing whether m, n, mn all almost perfect implies m, n powers of 2, and (2) trying the naive divisor bound to resolve this. Through sheer dumb luck this happened to work out perfectly. I thought this was kinda cool but I felt that I hadn't really unlocked a lot of the potential this idea had: then I basically tried to find the "general situation" which allows for this manipulation, and was amazed that it led to such a striking statement.