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Twitch Solves ISL

Episode 50

Problem

For every positive integer N, let $\sigma(N)$ denote the sum of the positive integer divisors of N. Find all integers $m \geq n \geq 2$ satisfying

$$\frac{\sigma(m)-1}{m-1}=\frac{\sigma(n)-1}{n-1}=\frac{\sigma(mn)-1}{mn-1}.$$

Video

https://youtu.be/jsw3c3yAn7o

Solution

The answer is that m and n should be powers of the same prime number. These all work because for a prime power we have

$$\frac{\sigma(p^e)-1}{p^e-1} = \frac{(1+p+\cdots+p^e)-1}{p^e-1} = \frac{p(1+\cdots+p^{e-1})}{p^e-1} = \frac{p}{p-1}.$$

So we now prove these are the only ones. Let λ be the common value of the three fractions.

Claim. Any solution (m, n) should satisfy $d(mn) = d(m) + d(n) - 1$.

Proof. The divisors of mn include the divisors of m, plus m times the divisors of n (counting m only once). Let λ be the common value; then this gives

$$\begin{aligned}\sigma(mn) &\geq \sigma(m) + m\sigma(n) - m \\ &= (\lambda m - \lambda + 1) + m(\lambda n - \lambda + 1) - m \\ &= \lambda mn - \lambda + 1\end{aligned}$$

and so equality holds. Thus these are all the divisors of mn, for a count of $d(m)+d(n)-1$. □

Claim. If $d(mn) = d(m) + d(n) - 1$ and $\min(m, n) \geq 2$, then m and n are powers of the same prime.

Proof. Let A denote the set of divisors of m and B denote the set of divisors of n. Then $|A \cdot B| = |A| + |B| - 1$ and $\min(|A|, |B|) > 1$, so $|A|$ and $|B|$ are geometric progressions with the same ratio. It follows that m and n are powers of the same prime. □

Remark (Nikolai Beluhov)**.** Here is a completion not relying on $|A \cdot B| = |A| + |B| - 1$. By the above arguments, we see that every divisor of mn is either a divisor of n, or n times a divisor of m.

Now suppose that some prime $p \mid m$ but $p \nmid n$. Then $p \mid mn$ but p does not appear in the above classification, a contradiction. By symmetry, it follows that m and n have the same prime divisors.

Now suppose we have different primes $p \mid m$ and $q \mid n$. Write $\nu_p(m) = \alpha$ and $\nu_p(n) = \beta$. Then $p^{\alpha+\beta} \mid mn$, but it does not appear in the above characterization, a contradiction. Thus, m and n are powers of the same prime.

Remark (Comments on the function in the problem)**.** Let $f(n) = \frac{\sigma(n)-1}{n-1}$. Then f is not really injective even outside the above solution; for example, we have $f(6 \cdot 11^k) = \frac{11}{5}$ for all k, plus sporadic equivalences like $f(14) = f(404)$, as pointed out by one reviewer during test-solving. This means that both relations should be used at once, not independently.

Remark (Authorship remarks)**.** Ankan gave the following story for how he came up with the problem while thinking about so-called *almost perfect* numbers.

> I was in some boring talk when I recalled a conjecture that if $\sigma(n) = 2n - 1$, then n is a power of 2. For some reason (divine intervention, maybe) I had the double idea of (1) seeing whether m, n, mn all almost perfect implies m, n powers of 2, and (2) trying the naive divisor bound to resolve this. Through sheer dumb luck this happened to work out perfectly. I thought this was kinda cool but I felt that I hadn't really unlocked a lot of the potential this idea had: then I basically tried to find the "general situation" which allows for this manipulation, and was amazed that it led to such a striking statement.