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Episode 50

Problem

Find all nonconstant polynomials $P(z)$ with complex coefficients for which all complex roots of the polynomials $P(z)$ and $P(z) - 1$ have absolute value 1.

Video

https://youtu.be/NOjZ_DKUgxc

External Link

https://aops.com/community/p20020202

Solution

The answer is $P(x)$ should be a polynomial of the form $P(x) = \lambda x^n - \mu$ where $|\lambda| = |\mu|$ and $\operatorname{Re}\mu = -\frac{1}{2}$. One may check these all work; let's prove they are the only solutions.

First approach (Evan Chen). We introduce the following notations:

$$\begin{aligned} P(x) &= c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \\ &= c_n(x+\alpha_1)\ldots(x+\alpha_n) \\ P(x) - 1 &= c_n(x+\beta_1)\ldots(x+\beta_n) \end{aligned}$$

By taking conjugates,

$$\begin{aligned} &(x+\alpha_1)\cdots(x+\alpha_n) = (x+\beta_1)\cdots(x+\beta_n) + c_n^{-1} \\ \implies &\left(x+\frac{1}{\alpha_1}\right)\cdots\left(x+\frac{1}{\alpha_n}\right) = \left(x+\frac{1}{\beta_1}\right)\cdots\left(x+\frac{1}{\beta_n}\right) + (\overline{c_n})^{-1} \qquad (\spadesuit) \end{aligned}$$

The equation ($\spadesuit$) is the main player:

Claim. We have $c_k = 0$ for all $k = 1, \ldots, n-1$.

Proof. By comparing coefficients of x^k in ($\spadesuit$) we obtain

$$\frac{c_{n-k}}{\prod_i \alpha_i} = \frac{c_{n-k}}{\prod_i \beta_i}$$

but $\prod_i \alpha_i - \prod_i \beta_i = \frac{1}{c_n} \neq 0$. Hence $c_k = 0$. $\square$

It follows that $P(x)$ must be of the form $P(x) = \lambda x^n - \mu$, so that $P(x) = \lambda x^n - (\mu+1)$. This requires $|\mu| = |\mu+1| = |\lambda|$ which is equivalent to the stated part.

Second approach (from the author). We let $A = P$ and $B = P - 1$ to make the notation more symmetric. We will as before show that A and B have all coefficients equal to zero other than the leading and constant coefficient; the finish is the same.

First, we rule out double roots.

Claim. Neither A nor B have double roots.

Proof. Suppose that b is a double root of B. By differentiating, we obtain $A' = B'$, so $A'(b) = 0$. However, by Gauss-Lucas, this forces $A(b) = 0$, contradiction. $\square$

Let $\omega = e^{2\pi i/n}$, let $a_1, \ldots, a_n$ be the roots of A, and let $b_1, \ldots, b_n$ be the roots of B. For each k, let A_k and B_k be the points in the complex plane corresponding to a_k and b_k.

Claim (Main claim)**.** For any i and j, $\frac{a_i}{a_j}$ is a power of ω.

Proof. Note that

$$\frac{a_i - b_1}{a_j - b_1} \cdots \frac{a_i - b_n}{a_j - b_n} = \frac{B(a_i)}{B(a_j)} = \frac{A(a_i) - 1}{A(a_j) - 1} = \frac{0-1}{0-1} = 1.$$

Since the points A_i, A_j, B_k all lie on the unit circle, interpreting the left-hand side geometrically gives

$$\measuredangle A_i B_1 A_j + \cdots + \measuredangle A_i B_n A_j = 0 \implies n\widehat{A_i A_j} = 0,$$

where angles are directed modulo 180° and arcs are directed modulo 360°. This implies that $\frac{a_i}{a_j}$ is a power of ω. $\square$

Now the finish is easy: since $a_1, \dots, a_n$ are all different, they must be $a_1\omega^0, \dots, a_1\omega^{n-1}$ in some order; this shows that A is a multiple of $x^n - a_1^n$, as needed.