

USAMO 1998/1

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TWITCH SOLVES ISL

Episode 49

Problem

Suppose that the set $\{1, 2, \dots, 1998\}$ has been partitioned into disjoint pairs $\{a_i, b_i\}$ ($1 \leq i \leq 999$) so that for all i , $|a_i - b_i|$ equals 1 or 6. Prove that the sum

$$|a_1 - b_1| + |a_2 - b_2| + \dots + |a_{999} - b_{999}|$$

ends in the digit 9.

Video

<https://youtu.be/jsw3c3yAn7o>

Solution

Let S be the sum. Modulo 2,

$$S = \sum |a_i - b_i| \equiv \sum (a_i + b_i) = 1 + 2 + \cdots + 1998 \equiv 1 \pmod{2}.$$

Modulo 5,

$$S = \sum |a_i - b_i| = 1 \cdot 999 \equiv 4 \pmod{5}.$$

So $S \equiv 9 \pmod{10}$.