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Twitch Solves ISL

Episode 49

Problem

Let T_n denotes the least positive integer such that

$$n \mid 1 + 2 + 3 + \cdots + T_n = \sum_{i=1}^{T_n} i.$$

Find all positive integers m such that $m \geq T_m$.

Video

https://youtu.be/l18c28zosHk

Solution

Fix an integer m and consider the equation

$$x(x+1) \equiv 0 \pmod{2m}.$$

The integer T_m is the smallest positive solution in x to this equation.

The residues $x \equiv 0 \pmod{2m}$ and $x \equiv 2m-1 \pmod{2m}$ are always solutions to this equation; we call these *trivial* solutions. Note more generally that:

Claim. If r is a solution, then so is $(2m-1)-r$.

Proof. Clear. □

This immediately implies that as long as there is a nontrivial solution, then $T_m \le \frac{1}{2}(2m-1)$, because the nontrivial solutions come in pairs with sum $2m-1$.

Now:

- If m is a power of 2, these are indeed the only solutions. So actually $T_m = 2m-1$ when m is a power of 2.
- In any other case, the Chinese remainder theorem implies the number of solutions is 2^k where k is the number of primes dividing $2m$ (without multiplicity). As $2^k > 2$, it follows there are nontrivial solutions, so these m do not work.