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Twitch Solves ISL

Episode 49

Problem

Let $\omega = e^{\frac{2\pi i}{5}}$ be a primitive fifth root of unity. Prove that there do not exist integers a, b, c, d, k with $k > 1$ such that

$$(a + b\omega + c\omega^2 + d\omega^3)^k = 1 + \omega.$$

Video

https://youtu.be/DzEdbFcr4DU

Solution

Preliminary definitions:

- Let $\varphi = \frac{1}{2}(1+\sqrt{5})$ and recall that $\operatorname{Re}\omega = \cos 72^\circ = \frac{\sqrt{5}-1}{4} = \frac{1}{2}(\varphi - 1)$.
- Let $K = \mathbb{Q}(\omega)$ and $L = \mathbb{Q}(\sqrt{5})$, so $\mathcal{O}_K = \mathbb{Z}[\omega]$ while $\mathcal{O}_L = \mathbb{Z}[\varphi]$.
- The problem is essentially asking to show that $1+\omega$ is not perfect power in $\mathcal{O}_K$.

We consider the following norm map:

$$N \colon \mathbb{Q}(\omega) \to \mathbb{Q}(\sqrt{5}) \text{ by } z \mapsto |z|^2 .$$

It restricts to a multiplicative map from $N \colon \mathcal{O}_K \to \mathcal{O}_L$. So we compute

$$N(z)^k = N(z^k) = N(1+\omega) = (1+\omega)\left(1+\frac{1}{\omega}\right) = 2 + 2\operatorname{Re}\omega = \varphi^2$$

which is in fact a unit in $\mathcal{O}_L$. On the other hand, it is known that the units of $\mathcal{O}_L$ are

$$\mathcal{O}_L^\times = \langle \varphi \rangle .$$

So this already means $k \le 2$, and the problem is reduced to proving the following claim.

Claim. There is no $z \in \mathcal{O}_K$ for which $N(z) = |z|^2 = \pm\varphi$.

Proof. Assume for contradiction there is such a z. We pick a different basis than the one given in the original problem statement, and assume instead that

$$z = a\omega + b\omega^2 + c\omega^3 + d\omega^4 \in \mathcal{O}_K$$

(which is nicer because it is more symmetric). Then

$$\begin{aligned} N(z) &= (a\omega + b\omega^2 + c\omega^3 + d\omega^4)(a\omega^{-1} + b\omega^{-2} + c\omega^{-3} + d\omega^{-4}) \\ &= (a^2+b^2+c^2+d^2) + (ab+bc+cd)\varphi + (ac+bd+ad)\varphi^2 \\ &= (a^2+b^2+c^2+d^2+ac+bd+ad) \\ &\quad + (ab+bc+cd+ac+bd+ad)\varphi \end{aligned}$$

However, this means

$$0 = a^2+b^2+c^2+d^2+ac+bd+ad = \frac{(a+c)^2}{2} + \frac{(b+d)^2}{2} + \frac{(a+d)^2}{2} + \frac{b^2+c^2}{2}$$

and this means $a=b=c=d=0$, a clear impossibility. $\square$