

# MR J489

Evan Chen

TWITCH SOLVES ISL

Episode 48

## Problem

Prove that in any triangle  $ABC$ , we have

$$\begin{aligned} 8r(R-2r)\sqrt{r(16R-5r)} &\leq a^3 + b^3 + c^3 - 3abc \\ &\leq 8R(R-2r)\sqrt{(2R+r)^2 + 2r^2}. \end{aligned}$$

## Video

[https://youtu.be/peeZTuB\\_cvY](https://youtu.be/peeZTuB_cvY)

## Solution

Let  $x = s - a$ ,  $y = s - b$ ,  $z = s - c$ , so  $x + y + z = s$  and  $a = y + z$  etc. A calculation gives

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= \frac{1}{2}(a + b + c) [(a - b)^2 + (b - c)^2 + (c - a)^2] \\ &= (x + y + z) [(x - y)^2 + (y - z)^2 + (z - x)^2] \\ &= 2(x^3 + y^3 + z^3 - 3xyz) \\ r &= \frac{[ABC]}{s} = \sqrt{\frac{xyz}{x + y + z}} \\ R &= \frac{(x + y)(y + z)(z + x)}{4\sqrt{xyz(x + y + z)}} = \underbrace{\frac{(x + y)(y + z)(z + x)}{4xyz}}_{k \geq 2} \cdot r. \end{aligned}$$

Define  $k = \frac{(x+y)(y+z)(z+x)}{4xyz} \geq 2$ .

**Left-hand inequality:** It can be written as

$$r^3 \cdot 4(k - 2) \cdot \sqrt{16k - 5} \leq x^3 + y^3 + z^3 - 3xyz.$$

By Schur's inequality, we have

$$0 \leq k - 2 = \frac{(x + y)(y + z)(z + x) - 8xyz}{4xyz} \leq \frac{x^3 + y^3 + z^3 - 3xyz}{4xyz}$$

Therefore it is enough to show

$$2r^3 \cdot \sqrt{16k - 5} \leq xyz \iff \sqrt{16k - 5} \cdot \sqrt{xyz} \leq (x + y + z)^{\frac{3}{2}}.$$

Rearranging, this is

$$\begin{aligned} \sqrt{4(x + y)(y + z)(z + x) - 5xyz} &\leq (x + y + z)^{3/2} \\ \iff 4 \sum_{\text{sym}} x^2y + 3xyz &\leq (x^3 + y^3 + z^3) + 3 \sum_{\text{sym}} x^2y + 6xyz \\ \iff \sum_{\text{sym}} x^2y &\leq (x^3 + y^3 + z^3) + 3xyz. \end{aligned}$$

which is also Schur.

**Right-hand inequality:** It can be written as

$$x^3 + y^3 + z^3 - 3xyz \leq r^3 \cdot 4k(k - 2)\sqrt{4k^2 + 4k + 3}.$$

We need the following intermediate claim.

**Claim.** We have  $x^3 + y^3 + z^3 - 3xyz \leq (x + y)(y + z)(z + x) \cdot (k - 2)$ .

*Proof.* Since  $k - 2 = \frac{(x+y)(y+z)(z+x) - 8xyz}{4xyz}$ , it is equivalent to prove that

$$4xyz(x^3 + y^3 + z^3 - 3xyz) + 8xyz(x + y)(y + z)(z + x) \leq (x + y)^2(y + z)^2(z + x)^2.$$

After full expansion, this becomes equivalent to

$$\sum_{\text{sym}} (x^4y^2 - x^4yz + x^3y^3 - 2x^3y^2z + (xyz)^2) \geq 0$$

or, in more legible triangle notation,

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & 0 & & 0 \\
 & & & 1 & -2 & & 1 \\
 & & 2 & -2 & -2 & 2 & \\
 & 1 & -2 & 6 & -2 & 1 & \\
 0 & -2 & -2 & -2 & -2 & 0 & \\
 0 & 0 & 1 & 2 & 1 & 0 & 0
 \end{array}$$

This is the sum of upsidedown triangle Schur (highlighted red) and simple AM-GM, i.e. by summing

$$\sum_{\text{sym}} (x^3y^3 - 2x^3y^2z + (xyz)^2) \geq 0$$

$$\sum_{\text{sym}} (x^4y^2 - x^4yz) \geq 0. \quad \square$$

Armed with the lemma, we reduce the inequality to

$$\begin{aligned}
 (x+y)(y+z)(z+x) &\leq r^3 \cdot 4k \sqrt{4k^2 + 4k + 3} \\
 \iff (x+y)(y+z)(z+x) &\leq r^3 \cdot 4 \cdot \frac{(x+y)(y+z)(z+x)}{4xyz} \sqrt{4k^2 + 4k + 3} \\
 \iff xyz &\leq \sqrt{\frac{(xyz)^3}{(x+y+z)^3}} \cdot \sqrt{4k^2 + 4k + 3} \\
 \iff (x+y+z)^3 &\leq xyz(4k^2 + 4k + 3) \\
 &= \frac{(x+y)^2(y+z)^2(z+x)^2}{4xyz} + (x+y)(y+z)(z+x) + 3xyz \\
 \iff x^3 + y^3 + z^3 + xyz + 2 \sum_{\text{sym}} x^2y &\leq \frac{(x+y)^2(y+z)^2(z+x)^2}{4xyz} = k \cdot (x+y)(y+z)(z+x).
 \end{aligned}$$

This is actually the same as the lemma, so we are done.

**Remark.** Alternatively, one could use so-called Gerretsen inequality that

$$r(16R - 5r) \leq s^2 \leq (2R + r)^2 + 2r^2$$

which sort of trivializes the problem, but only if one happens to know this theorem.