

# IMO 1997/5

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TWITCH SOLVES ISL

Episode 48

## Problem

Find all pairs  $(a, b)$  of positive integers satisfying

$$a^{b^2} = b^a.$$

## Video

<https://youtu.be/8UqGUia33mI>

## Solution

The answer is  $(1, 1)$ ,  $(16, 2)$  and  $(27, 3)$ .

We assume  $a, b > 1$  for convenience. Let  $T$  denote the set of non perfect powers other than 1.

**Claim.** Every integer greater than 1 is uniquely of the form  $t^n$  for some  $t \in T$ ,  $n \in \mathbb{N}$ .

*Proof.* Clear. □

Let  $a = s^m$ ,  $b = t^n$ .

$$s^{m \cdot (t^n)^2} = t^{n \cdot s^m}.$$

Hence  $s = t$  and we have

$$m \cdot t^{2n} = n \cdot t^m \implies t^{2n-m} = \frac{n}{m}.$$

Let  $n = t^e m$  and  $2 \cdot t^e m - m = e$ , or

$$e + m = 2t^e \cdot m.$$

We resolve this equation by casework

- If  $e > 0$ , then  $2t^e \cdot m > 2e \cdot m > e + m$ .
- If  $e = 0$  we have  $m = n$  and  $m = 2m$ , contradiction.
- If  $e = -1$  we apparently have

$$\frac{2}{t} \cdot m = m - 1 \implies m = \frac{t}{t-2}$$

so  $(t, m) = (3, 3)$  or  $(t, m) = (4, 2)$ .

- If  $e = -2$  we apparently have

$$\frac{2}{t^2} \cdot m = m - 2 \implies m = \frac{2}{1 - 2/t^2} = \frac{2t^2}{t^2 - 2}.$$

This gives  $(t, m) = (2, 2)$ .

- If  $e \leq -3$  then let  $k = -e \geq 3$ , so the equation is

$$m - k = \frac{2m}{t^k} \iff m = \frac{k \cdot t^k}{t^k - 2} = k + \frac{2k}{t^k - 2}.$$

However, for  $k \geq 3$  and  $t \geq 2$ , we always have  $2k \leq t^k - 2$ , with equality only when  $(t, k) = (2, 3)$ ; this means  $m = 4$ , which is not a new solution.