IMO 1997/5

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Twitch Solves ISL

Episode 48

Problem

Find all pairs (a, b) of positive integers satisfying

$$a^{b^2} = b^a.$$

Video

https://youtu.be/8UqGUia33mI

External Link

https://aops.com/community/p3845

Solution

The answer is $(1, 1)$, $(16, 2)$ and $(27, 3)$.

We assume $a, b > 1$ for convenience. Let T denote the set of non perfect powers other than 1.

Claim. Every integer greater than 1 is uniquely of the form t^n for some $t \in T$, $n \in \mathbb{N}$.

Proof. Clear. $\square$

Let $a = s^m$, $b = t^n$.

$$s^{m \cdot (t^n)^2} = t^{n \cdot s^m}.$$

Hence $s = t$ and we have

$$m \cdot t^{2n} = n \cdot t^m \implies t^{2n-m} = \frac{n}{m}.$$

Let $n = t^e m$ and $2 \cdot t^e m - m = e$, or

$$e + m = 2t^e \cdot m.$$

We resolve this equation by casework

- If $e > 0$, then $2t^e \cdot m > 2e \cdot m > e + m$.
- If $e = 0$ we have $m = n$ and $m = 2m$, contradiction.
- If $e = -1$ we apparently have

  $$\frac{2}{t} \cdot m = m - 1 \implies m = \frac{t}{t-2}$$

  so $(t, m) = (3, 3)$ or $(t, m) = (4, 2)$.
- If $e = -2$ we apparently have

  $$\frac{2}{t^2} \cdot m = m - 2 \implies m = \frac{2}{1 - 2/t^2} = \frac{2t^2}{t^2 - 2}.$$

  This gives $(t, m) = (2, 2)$.
- If $e \le -3$ then let $k = -e \ge 3$, so the equation is

  $$m - k = \frac{2m}{t^k} \iff m = \frac{k \cdot t^k}{t^k - 2} = k + \frac{2k}{t^k - 2}.$$

  However, for $k \ge 3$ and $t \ge 2$, we always have $2k \le t^k - 2$, with equality only when $(t, k) = (2, 3)$; this means $m = 4$, which is not a new solution.