CMIMC 2017 G9

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Twitch Solves ISL

Episode 48

Problem

Let $\triangle ABC$ be an acute triangle with circumcenter O, and let $Q \neq A$ denote the point on $\odot(ABC)$ for which $AQ \perp BC$. The circumcircle of $\triangle BOC$ intersects lines AC and AB for the second time at D and E respectively. Suppose that AQ, BC, and DE are concurrent. If $OD = 3$ and $OE = 7$, compute AQ.

Video

https://youtu.be/-beq_Y_Npsw

Solution

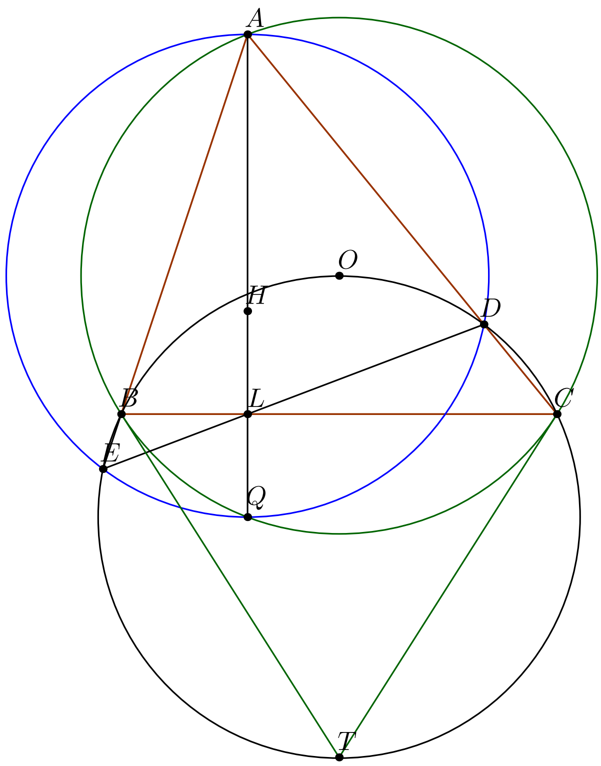
Let L be the concurrency point, so by radical axis $AEQD$ is cyclic.

Claim. Line DLE is the Simson line from Q to ABC.

Proof. Note

$$\measuredangle BED = \measuredangle BCD = \measuredangle BCA = \measuredangle BQA = \measuredangle BQL$$

so $BEQL$ is cyclic, hence $\measuredangle QEB = \measuredangle QLB = 90^\circ$, as desired. $\square$



Claim. $OEQD$ is a parallelogram.

Proof. Since $\measuredangle OED = \measuredangle OCD = 90^\circ - A$, we find $\overline{OE} \perp \overline{AC}$. But $\overline{QD} \perp \overline{AC}$ too. $\square$

The parallelogram law now gives

$$DE^2 + OQ^2 = 2(OD^2 + OE^2).$$

Also since O is the orthocenter of $\triangle ADE$ it is known that

$$AO^2 = AQ^2 - DE^2.$$

Putting these two together and noting $AO = OQ$ gives $AQ^2 = 2(OD^2 + OE^2) = 116$, hence $AQ = 2\sqrt{29}$.