# USAMO 1998/6 

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## Twitch Solves ISL

Episode 47

## Problem

Let $n \geq 5$ be an integer. Find the largest integer $k$ (as a function of $n$ ) such that there exists a convex $n$-gon $A_{1} A_{2} \ldots A_{n}$ for which exactly $k$ of the quadrilaterals $A_{i} A_{i+1} A_{i+2} A_{i+3}$ have an inscribed circle, where indices are taken modulo $n$.

## Video

https://youtu.be/yogTspROyC4

## External Link

https://aops.com/community/p150148

## Solution

The main claim is the following:
Claim. We can't have both $A_{1} A_{2} A_{3} A_{4}$ and $A_{2} A_{3} A_{4} A_{5}$ be circumscribed.
Proof. If not, then we have the following diagram, where $a=A_{1} A_{2}, b=A-2 A_{3}$, $c=A_{3} A_{4}, d=A_{4} A_{5}$.


Then $A_{1} A_{4}=c+a-b$ and $A_{5} A_{2}=b+d-c$. But now

$$
A_{1} A_{4}+A_{2} A_{5}=(c+a-b)+(b+d-c)=a+d=A_{1} A_{2}+A_{4} A_{5}
$$

but in the picture we have an obvious violation of the triangle inequality.
This immediately gives an upper bound of $\lfloor n / 2\rfloor$.
For the construction, one can construct a suitable cyclic $n$-gon by using a continuity argument (details to be added).

