USAMO 1998/6

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Twitch Solves ISL

Episode 47

Problem

Let $n \geq 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1 A_2 \dots A_n$ for which exactly k of the quadrilaterals $A_i A_{i+1} A_{i+2} A_{i+3}$ have an inscribed circle, where indices are taken modulo n.

Video

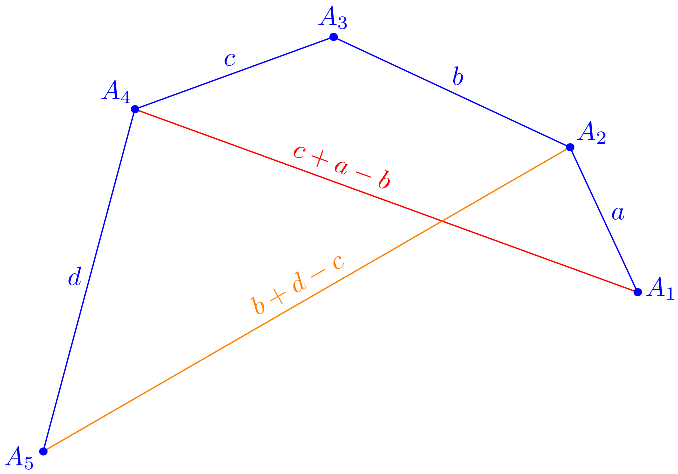
https://youtu.be/yogTspR0yC4

Solution

The main claim is the following:

Claim. We can't have both $A_1A_2A_3A_4$ and $A_2A_3A_4A_5$ be circumscribed.

Proof. If not, then we have the following diagram, where $a = A_1A_2$, $b = A - 2A_3$, $c = A_3A_4$, $d = A_4A_5$.



Then $A_1A_4 = c + a - b$ and $A_5A_2 = b + d - c$. But now

$$A_1A_4 + A_2A_5 = (c + a - b) + (b + d - c) = a + d = A_1A_2 + A_4A_5$$

but in the picture we have an obvious violation of the triangle inequality. □

This immediately gives an upper bound of $\lfloor n/2 \rfloor$.

For the construction, one can construct a suitable cyclic n-gon by using a continuity argument (details to be added).