Twitch 047.2

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Twitch Solves ISL

Episode 47

Problem

Solve over integers:

$$f(a-1)^2 + f(b-1)^2 = 1 - 2f(a)f(b) + f(a+b)^2.$$

Video

https://youtu.be/tfJRNQMya0c

Solution

The answers are $f(x) = x+1$, $f(x) = \begin{cases} 1 & x \text{ even} \\ 0 & x \text{ odd} \end{cases}$, and the negations of these two functions. One may check they work, so let's prove they are the only ones.

Let $P(a,b)$ denote the assertion. Then $P(a,-1)$ gives

$$\begin{aligned} f(a-1)^2 + f(-2)^2 &= 1 - 2f(-1)f(a) + f(a-1)^2 \\ \iff f(-2)^2 &= 1 + 2f(-1)f(a) \end{aligned}$$

So either $f(-1) = 0$ or f is constant (which is not possible).

Let's assume $f(-1) = 0$ then. Consider $P(0,0)$ to get $f(0)^2 = 1$. Since f works if and only $-f$ does, it is enough to tackle $f(0) = +1$. Then $P(a,0)$ gives

$$f(a-1)^2 = 1 - 2 \cdot f(a) + f(a)^2 = (1 - f(a))^2 .$$

We now consider two cases:

- If $f(1) = 2$: if we assume that $f(b-1) = b$ and $f(b) = b+1$,

$$\begin{aligned} 1 + f(b-1)^2 &= 1 - 4f(b) + f(b+1)^2 \\ f(b+1)^2 &= (b+2)^2 \\ f(b+2) &\in \{b, b+2\}. \end{aligned}$$

  So $f(x) = x+1$ works for any $x \geq 0$ by induction.

  To go downwards, we can get $f(a-1)$ for $a < 0$ by induction, by selecting large positive b.

- If $f(1) = 0$ then

$$1 + f(b-1)^2 = 1 + f(b+1)^2 \implies f(b-1)^2 = f(b+1)^2.$$

  This means all odd inputs are zero and all even inputs are ± 1.

  Plug in a and b even to get $f(a)f(b) = +1$, so f is constant on even inputs, as desired.