Shortlist 1990/17

Evan Chen

Twitch Solves ISL

Episode 47

Problem

Unit cubes are made into beads by drilling a hole through them along a diagonal. The beads are put on a string in such a way that they can move freely in space under the restriction that the vertices of two neighboring cubes are touching. Let A be the beginning vertex and B be the end vertex. Let there be pqr cubes on the string.

(a) Determine for which values of $p, q, r \geq 1$, it is possible to build a block with dimensions $p \times q \times r$.

(b) The same question as (a) with the extra condition that $A = B$.

Video

https://youtu.be/UWBYzWeJwYc

Solution

For (a), the answer is always yes; whereas for (b), the answer is only if at least two of p, q, r are even.

Let the box $\mathcal{B}$ be located in space spanning $(0,0,0)$ to (p,q,r). Then the string can move from (x,y,z) to $(x\pm 1, y\pm 1, z\pm 1)$. So, the string is going to visit only points of two parity classes in $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$, say $(\varepsilon_x \bmod 2, \varepsilon_y \bmod 2, \varepsilon_z \bmod 2)$ and $(1+\varepsilon_x \bmod 2, 1+\varepsilon_y \bmod 2, 1+\varepsilon_z \bmod 2)$. However, every unit cube has a string through it, so we find that the string visits every point of this form in $\mathcal{B}$. Let us denote the set of points by V (which depends on the choice of $(\varepsilon_x, \varepsilon_y, \varepsilon_z)$).

Hence the question is asking when we can have a Eulerian path or circuit on the resulting graph G (whose vertex set is V and whose edges are the string).

However, the degrees of vertices of G are always even *except* for the corners of $\mathcal{B}$, i.e. the eight points

$$\Big\{(0,0,0),\ (p,0,0),\ (0,q,0),\ (0,0,r),\ (p,q,0),\ (p,0,r),\ (0,q,r),\ (p,q,r)\Big\}.$$

- We can always ensure there are at most two corners in V by choosing ε_x, ε_y, ε_z randomly; the expected number of corners is $\frac{1}{4}\cdot 8 = 2$.

- We can ensure there are zero corners if at least two of (p,q,r) are even; otherwise we cannot.

Hence by the classical theorem on Eulerian paths and circuits, (a) is possible always, (b) is possible when at least two of (p,q,r) are even.