

PUMaC 2015 C8

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TWITCH SOLVES ISL

Episode 47

Problem

In a tournament with 2015 teams, each team plays every other team exactly once and no ties occur. Such a tournament is imbalanced if for every group of 6 teams, there exists either a team that wins against the other 5 or a team that loses to the other 5. If the teams are indistinguishable, what is the number of distinct imbalanced tournaments that can occur?

Video

https://youtu.be/YuGZ_XHtL8

Solution

We need the following lemma:

Lemma (Twitch Lemma). In a tournament if there is a $(k + 1)$ -cycle then there is a k -cycle.

Claim. A tournament is balanced if there exists two disjoint directed cycles, or a directed cycle of length at least 6.

Proof. In the former case, the Twitch Lemma gives us two directed triangles, which is bad.

In the latter case, the Twitch Lemma gives us a directed 6-cycle, which is bad. \square

Using this one can show (many details omitted) that the imbalanced tournaments are given by the following list:

- One fully ordered tournament
- $2013 \cdot 1$ tournaments that consist of a directed 3-cycle and everything else being fully ordered.
- $2012 \cdot 1$ tournaments that consist of a directed 4-cycle and everything else being fully ordered.
- $2011 \cdot 6$ tournament that consist of a directed 5-cycle and everything else being fully ordered.

This gives an answer of $1 + 2013 + 2012 + 2011 \cdot 6 = 16092$.