CAMO 2020/5

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Twitch Solves ISL

Episode 47

Problem

Let $f(x) = x^2 - 2$. Prove that for all positive integers n, the polynomial

$$P(x) = \underbrace{f(f(\dots f}_{n \text{ times}}(x)\dots)) - x$$

can be factored into two polynomials with integer coefficients and equal degree.

Video

https://youtu.be/ASXM4IsPyic

Solution

Note that for each $z \in \mathbb{C}$, we inductively have

$$P(2\cos z) = 2\left[\cos(2^n z) - \cos z\right].$$

We can now identify all the roots: this polynomial has roots at $\cos z$ for

$$z = 0,\ z = \frac{2\pi k}{2^n - 1},\ z = \frac{2\pi k}{2^n + 1}$$

for all integers k.

The 2^{n-1} roots of the form $2\cos\left(\frac{2\pi k}{2^n+1}\right)$ for $k = 1, \dots, 2^{n-1}$ can be used to form the polynomial

$$F = \prod_{k=1}^{2^{n-1}} \left(X - (\zeta^k + \zeta^{-k})\right) \qquad \text{where } \zeta = e^{\frac{2\pi i}{2^n+1}}$$

which has degree 2^{n-1} and divides P.

I claim it F has integer coefficients. In fact, we let T denote the normalized $(2^n + 1)$'th Chebyshev polynomial which maps $2\cos\theta$ to $2\cos((2^n+1)z)$, then T has integer coefficients and

$$T - 1 = F^2 \cdot (X - 1).$$

Indeed every root of F, i.e. number of the form $2\cos\left(\frac{2\pi k}{2^n+1}\right)$ is not only a root of F, but in fact a double root (because $T - 1 \le 0$ for inputs in $(-1, 1)$). A degree counting argument then implies these are all the roots.