RSL 2018 N1

Evan Chen

Twitch Solves ISL

Episode 45

Problem

Determine all polynomials f with integer coefficients such that $f(p)$ is a divisor of $2^p - 2$ for every odd prime p.

Video

https://youtu.be/YlhamV8TwSc

Solution

The only answers are ± 1, ± 2, ± 3, ± 6, $\pm x$ and $\pm 2x$. These can be seen to work, so we prove they are the only ones.

Claim. For every odd prime p, $f(p)$ only could have factors of 2, 3, or p.

Proof. Suppose $q \mid f(p)$ where $q \geq 3$ is a prime other than p. Consider primes ℓ such that $\ell \equiv p \pmod q$ and $\ell \equiv q-2 \mod q-1$; such primes exist by Dirichlet. Then

$$q \mid f(\ell) \mid 2^\ell - 2 \equiv 2^{q-2} - 2 \equiv \frac{1}{2} - 2 \pmod q$$

and hence $q = 3$. □

Claim. If $f(0) \neq 0$, then f is constant and a divisor of 6.

Proof. Let $p \equiv 5 \pmod 6$ be a large prime not dividing $f(0)$. Then $f(p) \not\equiv 0 \pmod p$, so $f(p) \mid 2 \cdot 3^\infty$ by the previous claim.

However, $2^p - 2 \equiv 3 \pmod 9$ and $2^p - 2 \equiv 2 \pmod 4$, so $f(p) \mid 6$.

Since this relation holds for all sufficiently large primes $p \equiv 5 \pmod 6$, f must be constant, and the conclusion follows. □

On the other hand if $f(x)$ is divisible by x, redo problem with $f(x)/x$. We can repeat this until f is constant.

Hence the possible candidates are $f(x) = cx^e$ where $c \mid 6$. We can deduce $e = 1$ by simply letting $p = 5$, and the rule out $3x$ and $6x$ by letting $p = 3$. So the problem is solved.