# RSL 2018 N1 <br> Evan Chen 

Twitch Solves ISL
Episode 45

## Problem

Determine all polynomials $f$ with integer coefficients such that $f(p)$ is a divisor of $2^{p}-2$ for every odd prime $p$.

## Video

https://youtu.be/YlhamV8TwSc

## Solution

The only answers are $\pm 1, \pm 2, \pm 3, \pm 6, \pm x$ and $\pm 2 x$. These can be seen to work, so we prove they are the only ones.

Claim. For every odd prime $p, f(p)$ only could have factors of 2,3 , or $p$.
Proof. Suppose $q \mid f(p)$ where $q \geq 3$ is a prime other than $p$. Consider primes $\ell$ such that $\ell \equiv p(\bmod q)$ and $\ell \equiv q-2 \bmod q-1$; such primes exist by Dirichlet. Then

$$
q|f(\ell)| 2^{\ell}-2 \equiv 2^{q-2}-2 \equiv \frac{1}{2}-2 \quad(\bmod q)
$$

and hence $q=3$.
Claim. If $f(0) \neq 0$, then $f$ is constant and a divisor of 6 .
Proof. Let $p \equiv 5(\bmod 6)$ be a large prime not dividing $f(0)$. Then $f(p) \not \equiv 0(\bmod p)$, so $f(p) \mid 2 \cdot 3^{\infty}$ by the previous claim.

However, $2^{p}-2 \equiv 3(\bmod 9)$ and $2^{p}-2 \equiv 2(\bmod 4)$, so $f(p) \mid 6$.
Since this relation holds for all sufficiently large primes $p \equiv 5(\bmod 6), f$ must be constant, and the conclusion follows.

On the other hand if $f(x)$ is divisible by $x$, redo problem with $f(x) / x$. We can repeat this until $f$ is constant.

Hence the possible candidates are $f(x)=c x^{e}$ where $c \mid 6$. We can deduce $e=1$ by simply letting $p=5$, and the rule out $3 x$ and $6 x$ by letting $p=3$. So the problem is solved.

