IMO 1998/1

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Twitch Solves ISL

Episode 45

Problem

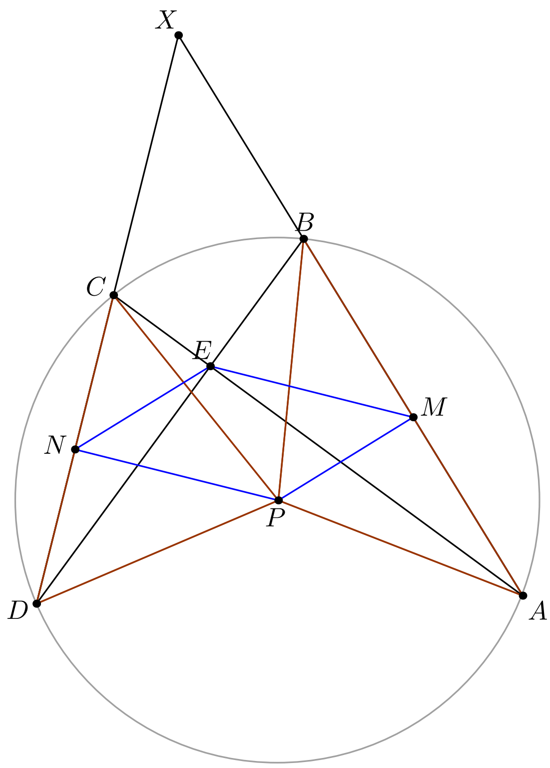
A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

Video

https://youtu.be/anCyZ5V0dgw

Solution

If $ABCD$ is cyclic, then P is the circumcenter, and $\angle APB + \angle PCD = 180^\circ$. The hard part is the converse.



Let M and N be the midpoints of $\overline{AB}$ and $\overline{CD}$.

Claim. Unconditionally, we have $\measuredangle NEM = \measuredangle MPN$.

Proof. Note that $\overline{EN}$ is the median of right triangle $\triangle ECD$, and similarly for $\overline{EM}$. Hence $\measuredangle NED = \measuredangle EDN = \measuredangle BDC$, while $\measuredangle AEM = \measuredangle ACB$. Since $\measuredangle DEA = 90^\circ$, by looking at quadrilateral $XDEA$ where $X = \overline{CD} \cap \overline{AB}$, we derive that $\measuredangle NED + \measuredangle AEM + \measuredangle DXA = 90^\circ$, so

$$\measuredangle NEM = \measuredangle NED + \measuredangle AEM + 90^\circ = -\measuredangle DXA = -\measuredangle NXM = -\measuredangle NPM$$

as needed. $\square$

However, the area condition in the problem tells us

$$\frac{EN}{EM} = \frac{CN}{CM} = \frac{PM}{PN}.$$

Finally, we have $\angle MEN > 90^\circ$ from the configuration. These properties uniquely determine the point E: it is the reflection of P across line MN.

So $EMPN$ is a parallelogram, and thus $\overline{ME} \perp \overline{CD}$. This implies $\measuredangle BAE = \measuredangle CEM = \measuredangle EDC$ giving $ABCD$ cyclic.