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Twitch Solves ISL

Episode 45

Problem

Let $n > 1$ be an integer. In a circular arrangement of n lamps $L_0, \dots, L_{n-1}$ each of of which can either ON or OFF and with indices taken modulo n, we start with the situation where all lamps are ON, and then carry out a sequence of steps. On the jth step (for $j = 0, 1, \dots$) if L_{j-1} is ON then L_j is toggled, whereas if L_{j-1} is OFF then nothing happens. Show that:

(i) There is a positive integer $M(n)$ such that after $M(n)$ steps all lamps are ON again,

(ii) If n has the form 2^k then all the lamps are ON after $n^2 - 1$ steps,

(iii) If n has the form $2^k + 1$ then all lamps are ON after $n^2 - n + 1$ steps.

Video

https://youtu.be/OvHiDzshu7M

Solution

For part (i), the sequence of states (and time indices modulo n) is finite, so it is certainly eventually periodic; however, because the operation is invertible, it is totally periodic.

Throughout the problem we always work modulo 2. By a *round* we mean a sequence of n consecutive operations. Note that a round has the following effect:

$$(a_1, \dots, a_{n-1}, a_0) \mapsto (a_2 + a_3 + \dots + a_0,\\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \dots,\ a_1 + a_2 + a_3 + \dots + a_0).$$

(We prefer to list L_1 first, rather than L_0.)

For (ii), we give a description of the process for $n = 2^{k+1}$ in terms of the process for $n = 2^k$. We write our states as binary strings of length n, with the ith bit corresponding to L_i (the last bit corresponds to $L_n = L_0$). The description goes as follows:

- We split the string into $L_1 L_2 \dots L_{n/2} \mid L_{n/2+1} \dots L_n$.
- There are two copies of the process for $n = 2^k$ side by side until we reach the string $100\cdots \mid 1000$.
- After the next round, we get $11\dots11 \mid 00\dots00$.
- Then, the left half plays another copy of the $n = 2^k$ process, with the right half being all zero, until we reach $1000 \mid 0000$
- Then $n-1$ turns later we get $111\dots111$ as needed.

For concreteness, if $n = 4$ is given by

$$1111 \xmapsto{\text{4 moves}} 1010 \xmapsto{\text{4 moves}} 1100 \xmapsto{\text{4 moves}} 1000 \xmapsto{\text{3 moves}} 1111$$

in terms of the rounds, then $n = 8$ is given by

$$\begin{aligned} 1111 \mid 1111 &\xmapsto{\text{8 moves}} 1010 \mid 1010 \xmapsto{\text{8 moves}} 1100 \mid 1100 \\ &\xmapsto{\text{8 moves}} 1000 \mid 1000 \xmapsto{\text{8 moves}} 1111 \mid 0000 \\ &\xmapsto{\text{8 moves}} 1010 \mid 0000 \xmapsto{\text{8 moves}} 1100 \mid 0000 \xmapsto{\text{8 moves}} 1000 \mid 0000. \end{aligned}$$

and seven moves later we return to all 1's. One can now check by a straightforward induction that the number of moves is $n^2 - 1$ exactly.

For (iii), we give a description for $n = 2^k + 1$ in terms of that for $n = 2^k$. All that needs to be observed is that the situation after the ith round of $n = 2^k + 1$ is the same as the ith round of $n = 2^k$, except with last bit moved to the first, and one extra 0 appended, up until we reach the special state $0010000\dots$. For example, for $n = 9$ we have

$$\begin{aligned} 111111111 &\xmapsto{\text{9 moves}} 001010101 \xmapsto{\text{9 moves}} 001100110 \\ &\xmapsto{\text{9 moves}} 001000100 \xmapsto{\text{9 moves}} 001111000 \\ &\xmapsto{\text{9 moves}} 001010000 \xmapsto{\text{9 moves}} 001100000 \xmapsto{\text{9 moves}} 001000000 \end{aligned}$$

and then after an additional $n + 1$ moves we return to all 1's. The total move count can be computed to be the desired $n^2 - n + 1$.