TSTST 2020/6

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Twitch Solves ISL

Episode 44

Problem

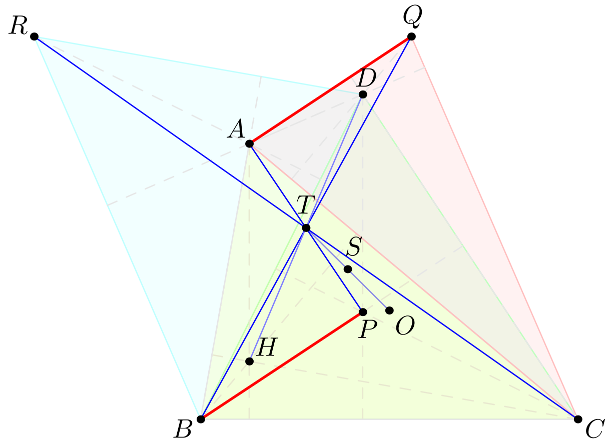
Let A, B, C, D be four points such that no three are collinear and D is not the orthocenter of triangle ABC. Let P, Q, R be the orthocenters of $\triangle BCD$, $\triangle CAD$, $\triangle ABD$, respectively. Suppose that lines AP, BQ, CR are pairwise distinct and are concurrent. Show that the four points A, B, C, D lie on a circle.

Video

https://youtu.be/FozF63KHw6E

Solution

Let T be the concurrency point, and let H be the orthocenter of $\triangle ABC$.



Claim (Key claim)**.** T is the midpoint of $\overline{AP}$, $\overline{BQ}$, $\overline{CR}$, $\overline{DH}$, and D is the orthocenter of $\triangle PQR$.

Proof. Note that $\overline{AQ} \parallel \overline{BP}$, as both are perpendicular to $\overline{CD}$. Since lines AP and BQ are distinct, lines AQ and BP are distinct.

By symmetric reasoning, we get that $AQCPBR$ is a hexagon with *opposite sides parallel* and *concurrent diagonals* as $\overline{AP}$, $\overline{BQ}$, $\overline{CR}$ meet at T. This implies that the *hexagon is centrally symmetric* about T; indeed

$$\frac{AT}{TP} = \frac{TQ}{BT} = \frac{CT}{TR} = \frac{TP}{AT}$$

so all the ratios are equal to $+1$.

Next, $\overline{PD} \perp \overline{BC} \parallel \overline{QR}$, so by symmetry we get D is the orthocenter of $\triangle PQR$. This means that T is the midpoint of $\overline{DH}$ as well. □

Corollary. The configuration is now symmetric: we have four points A, B, C, D, and their reflections in T are four orthocenters P, Q, R, H.

Let S be the centroid of $\{A, B, C, D\}$, and let O be the reflection of T in S. We are ready to conclude:

Claim. A, B, C, D are equidistant from O.

Proof. Let A', O', S', T', D' be the projections of A, O, S, T, D onto line BC.

Then T' is the midpoint of $\overline{A'D'}$, so $S' = \frac{1}{4}(A' + D' + B + C)$ gives that O' is the midpoint of $\overline{BC}$.

Thus $OB = OC$ and we're done. □