TSTST 2020/5

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Twitch Solves ISL

Episode 44

Problem

Let $\mathbb{N}^2$ denote the set of ordered pairs of positive integers. A finite subset S of $\mathbb{N}^2$ is *stable* if whenever (x, y) is in S, then so are all points (x', y') of $\mathbb{N}^2$ with both $x' \le x$ and $y' \le y$.

Prove that if S is a stable set, then among all stable subsets of S (including the empty set and S itself), at least half of them have an even number of elements.

Video

https://youtu.be/L_JBme8pnKU

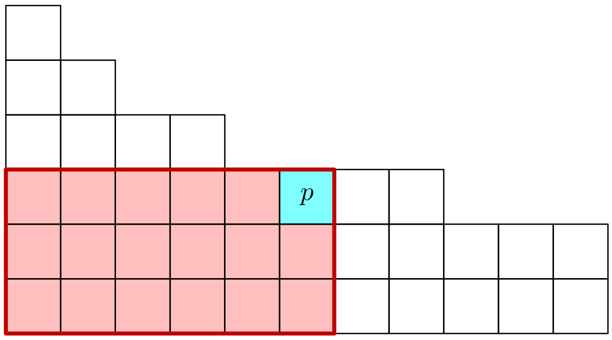
External Link

https://aops.com/community/p19444403

Solution

The following inductive solution was given by Nikolai Beluhov. We proceed by induction on $|S|$, with $|S| \le 1$ clear.

Suppose $|S| \ge 2$. For any $p \in S$, let $R(p)$ denote the stable rectangle with upper-right corner p. We say such p is *pivotal* if $p + (1,1) \notin S$ and $|R(p)|$ is even.



Claim. If $|S| \ge 2$, then a pivotal p always exists.

Proof. Consider the top row of S.

- If it has length at least 2, one of the two rightmost points in it is pivotal.
- Otherwise, the top row has length 1. Now either the top point or the point below it (which exists as $|S| \ge 2$) is pivotal. □

We describe how to complete the induction, given some pivotal $p \in S$. There is a partition

$$S = R(p) \sqcup S_1 \sqcup S_2$$

where S_1 and S_2 are the sets of points in S above and to the right of p (possibly empty).

Claim. The desired inequality holds for stable subsets containing p.

Proof. Let E_1 denote the number of even stable subsets of S_1; denote E_2, O_1, O_2 analogously. The stable subsets containing p are exactly $R(p) \sqcup T_1 \sqcup T_2$, where $T_1 \subseteq S_1$ and $T_2 \subseteq S_2$ are stable.

Since $|R(p)|$ is even, exactly $E_1E_2 + O_1O_2$ stable subsets containing p are even, and exactly $E_1O_2 + E_2O_1$ are odd. As $E_1 \ge O_1$ and $E_2 \ge O_2$ by inductive hypothesis, we obtain $E_1E_2 + O_1O_2 \ge E_1O_2 + E_2O_1$ as desired. □

By the inductive hypothesis, the desired inequality also holds for stable subsets not containing p, so we are done.