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TWITCH SOLVES ISL

Episode 43

Problem

There are some counters in some cells of a 100×100 board. A cell C is *nice* if there are an even number of counters in cells which share a side with C . Can exactly one cell be nice?

Video

<https://youtu.be/FozF63KHw6E>

External Link

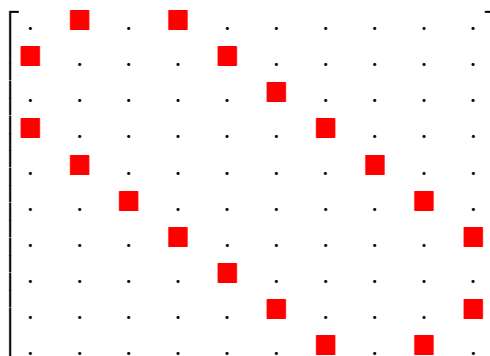
<https://aops.com/community/p2262408>

Solution

A *good set* is a set S of cells such that any cell is adjacent to an even number of cells in S .

Claim. For any cell C , there exists a good set S containing C with an even number of elements.

Proof. All sets of the following shape are good (red squares mark the points of S):



Descriptively, the cells look like a “tilted rectangle”. By varying the dimensions and orientation of the rectangle, we can cover any cell. (There is an edge case including just the major diagonal.) \square

If there is exactly one nice cell C , then consider the set S and consider

$$\sum_{\text{counter } x} \#(\text{cells in } S \text{ adjacent to } x).$$

On the one hand, it is even by the definition of good set. On the other hand, counting by the cells, the total should be odd, contradiction.

Remark. By pairing this result with the tiling from IMO 1999/3 or EGMO 2020/2, it follows that there also cannot be exactly one non-nice cell.