DeuxMO 2020/1/2.2

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Twitch Solves ISL

Episode 43

Problem

Solve $f(f(x)f(y)) = xy - x - y$ over reals.

Video

https://youtu.be/KXdg6Kv7d9k

Solution

The only answer is $t \mapsto t-1$ which obviously works.

Let $P(x, y)$ denote the given assertion.

Claim. f is a bijection.

Proof. Follows from $P(x, 0)$. □

Claim. $f(1) = 0$ and $f(0) = -1$.

Proof. From $P(x, 1)$ we have $f(f(x)f(1)) = -1$. Since f is bijection, this can only occur if $f(1) = 0$. □

By $P(x, 0)$ we have $f(-f(x)) = -x$. So, if we let $g(x) = -f(x)$, then g is an involution and

$$g(g(x)g(y)) = x + y - xy$$

or equivalently

$$\begin{aligned} g(xy) &= g(x) + g(y) - g(x) - g(y) \\ \iff 1 - g(xy) &= (1 - g(x))(1 - g(y)). \end{aligned}$$

Let's define

$$h = 1 - g$$

so h is multiplicative and bijective with $h(0) = 0$, $h(1) = 1$, and hence $h(-1) = -1$. Then the relation $g(g(x)) = x$ rewrites as

$$h(1 - h(x)) = 1 - x. \qquad (\heartsuit)$$

Claim. We have $h(h(x)) = x$.

Proof. Via

$$\begin{aligned} \frac{h(-1) \cdot (1-x)}{h(h(x))} &= \frac{h(-1) \cdot h(1 - h(x))}{h(h(x))} = \frac{h(h(x) - 1)}{h(h(x))} \\ &= h\left(1 - \frac{1}{h(x)}\right) = h\left(1 - h\left(\frac{1}{x}\right)\right) = 1 - \frac{1}{x}. \end{aligned}$$ □

Now $(\heartsuit)$ and the previous claim gives $1 - h(x) = h(1 - x)$; and by taking $x = a/b$, we find that h is an additive function. Since h is both additive and multiplicative on the real numbers (and nonconstant), it must be the identity function. This gives the set of solutions claimed earlier.