DeuX MO 2020/I/2

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Twitch Solves ISL

Episode 43

Problem

Solve $f(f(x)f(y)) = xy - x - y$ over reals.

Video

https://youtu.be/KXdg6Kv7d9k

Solution

The only answer is $t \mapsto t-1$ which obviously works.

Let $P(x,y)$ denote the given assertion.

Claim. f is a bijection.

Proof. Follows from $P(x,0)$. □

Claim. $f(1)=0$ and $f(0)=-1$.

Proof. From $P(x,1)$ we have $f(f(x)f(1))=-1$. Since f is bijection, this can only occur if $f(1)=0$. □

By $P(x,0)$ we have $f(-f(x))=-x$. So, if we let $g(x)=-f(x)$, then g is an involution and

$$g(g(x)g(y)) = x+y-xy$$

or equivalently

$$\begin{aligned} & g(xy) = g(x)+g(y)-g(x)-g(y) \\ \iff & 1-g(xy) = (1-g(x))(1-g(y)). \end{aligned}$$

Let's define

$$h = 1-g$$

so h is multiplicative and bijective with $h(0)=0$, $h(1)=1$, and hence $h(-1)=-1$. Then the relation $g(g(x))=x$ rewrites as

$$h(1-h(x)) = 1-x. \qquad (\heartsuit)$$

Claim. We have $h(h(x))=x$.

Proof. Via

$$\begin{aligned} \frac{h(-1)\cdot(1-x)}{h(h(x))} &= \frac{h(-1)\cdot h(1-h(x))}{h(h(x))} = \frac{h(h(x)-1)}{h(h(x))} \\ &= h\left(1-\frac{1}{h(x)}\right) = h\left(1-h\left(\frac{1}{x}\right)\right) = 1-\frac{1}{x}. \end{aligned}$$ □

Now $(\heartsuit)$ and the previous claim gives $1-h(x)=h(1-x)$; and by taking $x=a/b$, we find that h is an additive function. Since h is both additive and multiplicative on the real numbers (and nonconstant), it must be the identity function. This gives the set of solutions claimed earlier.