# TSTST 2011/2 

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## Twitch Solves ISL

Episode 42

## Problem

Two circles $\omega_{1}$ and $\omega_{2}$ intersect at points $A$ and $B$. Line $\ell$ is tangent to $\omega_{1}$ at $P$ and to $\omega_{2}$ at $Q$ so that $A$ is closer to $\ell$ than $B$. Let $X$ and $Y$ be points on major $\operatorname{arcs} \widehat{P A}$ (on $\left.\omega_{1}\right)$ and $\widehat{A Q}\left(\right.$ on $\left.\omega_{2}\right)$, respectively, such that $A X / P X=A Y / Q Y=c$. Extend segments $P A$ and $Q A$ through $A$ to $R$ and $S$, respectively, such that $A R=A S=c \cdot P Q$. Given that the circumcenter of triangle $A R S$ lies on line $X Y$, prove that $\angle X P A=\angle A Q Y$.

## Video

https://youtu.be/U5EgFWrDO-E

## External Link

https://aops.com/community/p2374843

## Solution

We begin as follows:
Claim. There is a spiral similarity centered at $X$ mapping $A R$ to $P Q$. Similarly there is a spiral similarity centered at $Y$ mapping $S A$ to $P Q$.

Proof. Since $\measuredangle X A R=\measuredangle X A P=\measuredangle X P Q$, and $A R / A X=P Q / P X$ is given.
Now the composition of the two spiral similarities

$$
A R \stackrel{X}{\longmapsto} P Q \stackrel{Y}{\mapsto} S A
$$

must be a rotation, since $A R=A S$. The center of this rotation must coincide with the circumcenter $O$ of $\triangle A R S$, which is known to lie on line $X Y$.


As $O$ is a fixed-point of the composed map above, we may let $O^{\prime}$ be the image of $O$ under the rotation at $X$, so that

$$
\triangle X P A \stackrel{+}{\sim} \triangle X O^{\prime} O, \quad \triangle Y Q A \stackrel{+}{\sim} \triangle Y O^{\prime} O .
$$

Because

$$
\frac{X O}{X O^{\prime}}=\frac{X A}{X P}=c \frac{Y Q}{Y A}=\frac{Y O}{Y O^{\prime}}
$$

it follows $\overline{O^{\prime} O}$ bisects $\angle X O^{\prime} Y$. Finally, we have

$$
\measuredangle X P A=\measuredangle X O^{\prime} O=\measuredangle O O^{\prime} Y=\measuredangle A Q Y
$$

Remark. Indeed, this also shows $\overline{X P} \| \overline{Y Q}$; so the positive homothety from $\omega_{1}$ to $\omega_{2}$ maps $P$ to $Q$ and $X$ to $Y$.

