TSTST 2011/2

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Twitch Solves ISL

Episode 42

Problem

Two circles ω_1 and ω_2 intersect at points A and B. Line ℓ is tangent to ω_1 at P and to ω_2 at Q so that A is closer to ℓ than B. Let X and Y be points on major arcs $\widehat{PA}$ (on ω_1) and AQ (on ω_2), respectively, such that $AX/PX = AY/QY = c$. Extend segments PA and QA through A to R and S, respectively, such that $AR = AS = c \cdot PQ$. Given that the circumcenter of triangle ARS lies on line XY, prove that $\angle XPA = \angle AQY$.

Video

https://youtu.be/U5EgFWrDO-E

Solution

We begin as follows:

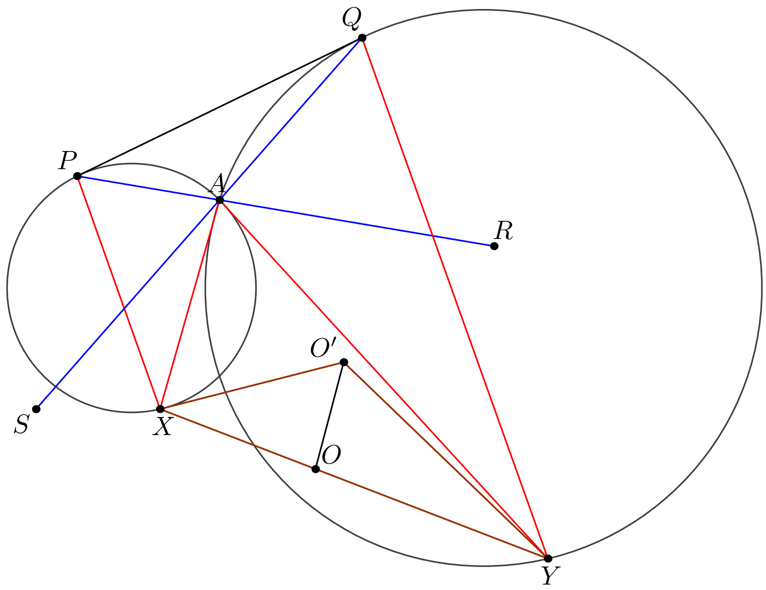
Claim. There is a spiral similarity centered at X mapping AR to PQ. Similarly there is a spiral similarity centered at Y mapping SA to PQ.

Proof. Since $\measuredangle XAR = \measuredangle XAP = \measuredangle XPQ$, and $AR/AX = PQ/PX$ is given. $\square$

Now the composition of the two spiral similarities

$$AR \overset{X}{\longmapsto} PQ \overset{Y}{\longmapsto} SA$$

must be a rotation, since $AR = AS$. The center of this rotation must coincide with the circumcenter O of $\triangle ARS$, which is known to lie on line XY.



Thus, we may let O' be the image under the rotation at X, so that

$$\triangle XPA \overset{+}{\sim} \triangle XO'O, \qquad \triangle YQA \overset{+}{\sim} \triangle YO'O.$$

Because

$$\frac{XO}{XO'} = \frac{XA}{XP} = c\frac{YQ}{YA} = \frac{YO}{YO'}$$

it follows $\overline{O'O}$ bisects $\angle XO'Y$. Finally, we have

$$\measuredangle XPA = \measuredangle XO'O = \measuredangle OO'Y = \measuredangle AQY.$$

Remark. Indeed, this also shows $\overline{XP} \parallel \overline{YQ}$; so the positive homothety from ω_1 to ω_2 maps P to Q and X to Y.