TSTST 2011/2 Evan Chen

TWITCH SOLVES ISL

Episode 42

Problem

Two circles ω_1 and ω_2 intersect at points A and B. Line ℓ is tangent to ω_1 at P and to ω_2 at Q so that A is closer to ℓ than B. Let X and Y be points on major arcs \widehat{PA} (on ω_1) and \widehat{AQ} (on ω_2), respectively, such that AX/PX = AY/QY = c. Extend segments PA and QA through A to R and S, respectively, such that $AR = AS = c \cdot PQ$. Given that the circumcenter of triangle ARS lies on line XY, prove that $\angle XPA = \angle AQY$.

Video

https://youtu.be/U5EgFWrDO-E

External Link

https://aops.com/community/p2374843

Solution

We begin as follows:

Claim. There is a spiral similarity centered at X mapping AR to PQ. Similarly there is a spiral similarity centered at Y mapping SA to PQ.

Proof. Since $\measuredangle XAR = \measuredangle XAP = \measuredangle XPQ$, and AR/AX = PQ/PX is given.

Now the composition of the two spiral similarities

$$AR \xrightarrow{X} PQ \xrightarrow{Y} SA$$

must be a rotation, since AR = AS. The center of this rotation must coincide with the circumcenter O of $\triangle ARS$, which is known to lie on line XY.



As O is a fixed-point of the composed map above, we may let O' be the image of O under the rotation at X, so that

$$\triangle XPA \stackrel{+}{\sim} \triangle XO'O, \qquad \triangle YQA \stackrel{+}{\sim} \triangle YO'O.$$

Because

$$\frac{XO}{XO'} = \frac{XA}{XP} = c\frac{YQ}{YA} = \frac{YO}{YO'}$$

it follows $\overline{O'O}$ bisects $\angle XO'Y$. Finally, we have

$$\measuredangle XPA = \measuredangle XO'O = \measuredangle OO'Y = \measuredangle AQY.$$

Remark. Indeed, this also shows $\overline{XP} \parallel \overline{YQ}$; so the positive homothety from ω_1 to ω_2 maps P to Q and X to Y.