Cono Sur 2020/6

Evan Chen

Twitch Solves ISL

Episode 42

Problem

A 4 x 4 square board is called *brasuca* if it follows all the conditions:

- each box contains one of the numbers $0, 1, 2, 3, 4$ or 5;
- the sum of the numbers in each row is 5;
- the sum of the numbers in each column is 5;
- the sum of the numbers on each diagonal of four squares is 5;
- the number written in the upper left box of the board is less than or equal to the other numbers the board;
- when dividing the board into four 2×2 squares, in each of them the sum of the four numbers and 5.

How many brasucas boards are there?

Video

https://youtu.be/iQkSQ97KWCA

Solution

Let $S = 5$.

Let's first consider the case where the upper-left number is 0. Then there is a bijection between ordered 6-tuples of nonnegative integers (a, b, c, d, x, y) with sum S, and brasuca boards, given as follows:

$$\begin{array}{cc|cc} 0 & a+b & c+x & d+y \\ c+d & x+y & \color{red}{a} & \color{red}{b} \\ \hline a+y & \color{red}{c} & b+d & \color{red}{x} \\ b+x & \color{red}{d} & \color{red}{y} & a+c \end{array}$$

Indeed, the six numbers written can be checked to uniquely determine the board; conversely, the board is obviously valid.

Hence, the number of boards with 0 in the upper-left hand corner is exactly $\binom{S+5}{5} = 252$ by sticks-and-stones.

For the case where the upper-left number is 1 is the same, one subtracts 1 from every number on the board to arrive at the same problem with $S = 1$ instead. The answer then is $\binom{S+5}{5} = 6$.

Hence the final answer is 258.

Remark. In general, if 5 is replaced by a target T, the answer would simply be

$$\binom{T+5}{5} + \binom{T+1}{5} + \binom{T-3}{5} + \cdots.$$