USAMO 1999/1

Evan Chen

Twitch Solves ISL

Episode 41

Problem

Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

(a) every square that does not contain a checker shares a side with one that does;

(b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2 - 2)/3$ checkers have been placed on the board.

Video

https://youtu.be/Uv4AtkRVPwo

External Link

https://aops.com/community/p340035

Solution

Take a spanning tree on the set V of checkers where the $|V|-1$ edges of the tree are given by orthogonal adjacency. By condition (a) we have

$$\sum_{v \in V} (4 - \deg v) \geq n^2 - |V|$$

and since $\sum_{v \in V} \deg v = 2(|V| - 1)$ we get

$$4|V| - (2|V| - 2) \geq n^2 - |V|$$

which implies $|V| \geq \frac{n^2-2}{3}$.