USAMO 1998/4

Evan Chen

Twitch Solves ISL

Episode 41

Problem

A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

Video

https://youtu.be/0_e8MuvtsJo

Solution

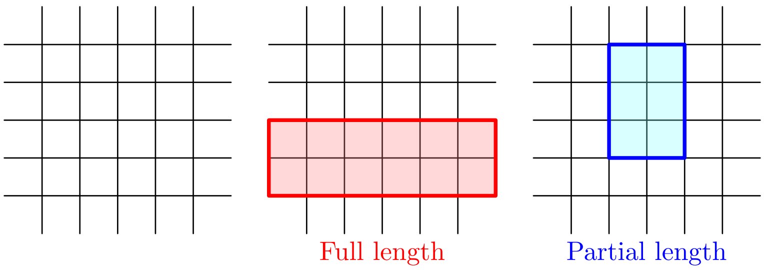
The answer is 98. One of several possible constructions is to toggle all columns and rows with even indices.

In the other direction, let $n = 98$ and suppose that k rectangles are used, none of which are $n \times n$ (else we may delete it). Then, for any two orthogonally adjacent cells, the edge between them must be contained in the edge of one of the k rectangles.

We define a *gridline* to be a line segment that runs in the interior of the board from one side of the board to the other. Hence there are $2n - 2$ gridlines exactly. Moreover, we can classify these rectangles into two types:

- *Full length rectangles*: these span from one edge of the board to the other. The two long sides completely cover two gridlines, but the other two sides of the rectangle do not.
- *Partial length rectangles*: each of four sides can partially cover "half a" gridlines.

See illustration below for $n = 6$.



Since there are $2n - 2$ gridlines; and each rectangle can cover at most two gridlines in total (where partial-length rectangles are "worth $\frac{1}{2}$" on each of the four sides), we immediately get the bound $2k \ge 2n - 2$, or $k \ge n - 1$.

To finish, we prove that:

Claim. If equality holds and $k = n - 1$, then n is odd.

Proof. If equality holds, then look at the horizontal gridlines and say two gridlines are *related* if some rectangle has horizontal edges along both gridlines. (Hence, the graph has degree either 1 or 2 at each vertex, for equality to hold.) The reader may verify the resulting graph consists only of even length cycles and single edges, which would mean $n - 1$ is even. □

Hence for $n = 98$ the answer is indeed 98 as claimed.