# USAMO 1998/4 Evan Chen

Twitch Solves ISL

Episode 41

### Problem

A computer screen shows a  $98 \times 98$  chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

# Video

https://youtu.be/0\_e8MuvtsJo

## **External Link**

https://aops.com/community/p343869

#### Solution

The answer is 98. One of several possible constructions is to toggle all columns and rows with even indices.

In the other direction, let n = 98 and suppose that k rectangles are used, none of which are  $n \times n$  (else we may delete it). Then, for any two orthogonally adjacent cells, the edge between them must be contained in the edge of one of the k rectangles.

We define a *gridline* to be a line segment that runs in the interior of the board from one side of the board to the other. Hence there are 2n - 2 gridlines exactly. Moreover, we can classify these rectangles into two types:

- *Full length rectangles*: these span from one edge of the board to the other. The two long sides completely cover two gridlines, but the other two sides of the rectangle do not.
- Partial length rectangles: each of four sides can partially cover "half a" gridline.

See illustration below for n = 6.



Since there are 2n - 2 gridlines; and each rectangle can cover at most two gridlines in total (where partial-length rectangles are "worth  $\frac{1}{2}$ " on each of the four sides), we immediately get the bound  $2k \ge 2n - 2$ , or  $k \ge n - 1$ .

To finish, we prove that:

**Claim.** If equality holds and k = n - 1, then n is odd.

*Proof.* If equality holds, then look at the horizontal gridlines and say two gridlines are *related* if some rectangle has horizontal edges along both gridlines. (Hence, the graph has degree either 1 or 2 at each vertex, for equality to hold.) The reader may verify the resulting graph consists only of even length cycles and single edges, which would mean n-1 is even.

Hence for n = 98 the answer is indeed 98 as claimed.