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Twitch Solves ISL

Episode 41

Problem

Let r be a rational number in the interval $[-1, 1]$ and let $\theta = \cos^{-1} r$. Call a subset S of the plane good if S is unchanged upon rotation by θ around any point of S (in both clockwise and counterclockwise directions). Determine all values of r satisfying the following property: The midpoint of any two points in a good set also lies in the set.

Video

https://youtu.be/NQe9616AqXU

External Link

https://aops.com/community/p3181487

Solution

The answer is that r has this property if and only if $r = \frac{4n-1}{4n}$ for some integer n.

Throughout the solution, we will let $r = \frac{a}{b}$ with $b > 0$ and $\gcd(a, b) = 1$. We also let

$$\omega = e^{i\theta} = \frac{a}{b} \pm \frac{\sqrt{b^2 - a^2}}{b} i.$$

This means we may work with complex multiplication in the usual way; the rotation of z through center c is given by $z \mapsto \omega(z - c) + c$.

For most of our proof, we start by constructing a good set as follows.

- Start by letting $S_0 = \{0, 1\}$.
- Let S_i consist of S_{i-1} plus all points that can be obtained by rotating a point of S_{i-1} through a different point of S_{i-1} (with scale factor ω).
- Let $S_\infty = \bigcup_{i \ge 0} S_i$.

The set S_∞ is the (minimal, by inclusion) good set containing 0 and 1. We are going to show that for most values of r, we have $\frac{1}{2} \notin S_\infty$.

Claim. If b is odd, then $\frac{1}{2} \notin S_\infty$.

Proof. Idea: denominators that appear are always odd.

Consider the ring

$$A = \mathbb{Z}_{\{b\}} = \left\{ \frac{s}{t} \mid s, t \in \mathbb{Z}, t \mid b^\infty \right\}$$

which consists of all rational numbers whose denominators divide b^∞. Then, $0, 1, \omega \in A[\sqrt{b^2 - a^2}]$ and hence $S_\infty \subseteq A[\sqrt{b^2 - a^2}]$ too. (This works even if $\sqrt{b^2 - a^2} \in \mathbb{Z}$, in which case $S_\infty \subseteq A = A[\sqrt{b^2 - a^2}]$.)

But $\frac{1}{2} \notin A[\sqrt{b^2 - a^2}]$. $\square$

Claim. If b is even and $|b - a| \neq 1$, then $\frac{1}{2} \notin S_\infty$.

Proof. Idea: take modulo a prime dividing $b - a$.

Let $D = b^2 - a^2 \equiv 3 \pmod 4$. Let p be a prime divisor of $b - a$ with odd multiplicity. Because $\gcd(a, b) = 1$, we have $p \neq 2$ and $p \nmid b$.

Consider the ring

$$A = \mathbb{Z}_{(p)} = \left\{ \frac{s}{t} \mid s, t \in \mathbb{Z}, p \perp t \right\}$$

which consists of all rational numbers whose denominators are coprime to p. Then, $0, 1, \omega \in A[\sqrt{-D}]$ and hence $S_\infty \subseteq A[\sqrt{-D}]$ too.

Now, there is a well-defined "mod-p" ring homomorphism

$$\Psi \colon A[\sqrt{-D}] \to \mathbb{F}_p \quad \text{by} \quad x + y\sqrt{-D} \mapsto x \bmod p$$

which commutes with addition and multiplication (as $p \mid D$). Under this map,

$$\omega \mapsto \frac{a}{b} \bmod p = 1.$$

Consequently, the rotation $z \mapsto \omega(z - c) + c$ is just the identity map modulo p. In other words, the pre-image of any point in S_∞ under Ψ must be either $\Psi(0) = 0$ or $\Psi(1) = 1$.

However, $\Psi(1/2) = 1/2$ is neither of these. So this point cannot be achieved. $\square$

Claim. Suppose $a = 2n - 1$ and $b = 2n$ for n an odd integer. Then $\frac{1}{2} \notin S_\infty$

Proof. Idea: ω is "algebraic integer" sans odd denominators.

This time, we define the ring

$$B = \mathbb{Z}_{(2)} = \left\{ \frac{s}{t} \mid s, t \in \mathbb{Z}, t \text{ odd} \right\}$$

of rational numbers with odd denominator. We carefully consider the ring $B[\omega]$ where

$$\omega = \frac{2n - 1 \pm \sqrt{1 - 4n}}{2n}.$$

So $S_\infty \subseteq B[\omega]$ as $0, 1, \omega \in B[\omega]$.

I claim that $B[\omega]$ is an integral extension of B; equivalently that ω is integral over B. Indeed, ω is the root of the monic polynomial

$$(T-1)^2 + \frac{1}{n}(T-1) - \frac{1}{n} = 0$$

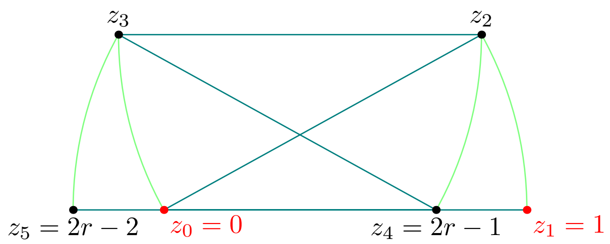
where $\frac{1}{n} \in B$ makes sense as n is odd.

On the other hand, $\frac{1}{2}$ is not integral over B so it is not an element of $B[\omega]$. $\square$

It remains to show that if $r = \frac{4n-1}{4n}$, then goods sets satisfy the midpoint property. Again starting from the points $z_0 = 0$, $z_1 = 1$ construct the sequence

$$\begin{aligned}
z_2 &= \omega(z_1 - z_0) + z_0 \\
z_3 &= \omega^{-1}(z_0 - z_2) + z_2 \\
z_4 &= \omega^{-1}(z_2 - z_3) + z_3 \\
z_5 &= \omega(z_3 - z_4) + z_4
\end{aligned}$$

as shown in the diagram below.



This construction shows that if we have the length-one segment $\{0, 1\}$ then we can construct the length-one segment $\{2r - 2, 2r - 1\}$. In other words, we can shift the segment to the left by

$$1 - (2r - 1) = 2(1 - r) = \frac{1}{2n}.$$

Repeating this construction n times gives the desired midpoint $\frac{1}{2}$.