USAMO 1998/3

Evan Chen

Twitch Solves ISL

Episode 40

Problem

Let $a_0, a_1, \cdots, a_n$ be numbers from the interval $(0, \pi/2)$ such that $\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \cdots + \tan(a_n - \frac{\pi}{4}) \geq n - 1$. Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}.$$

Video

https://youtu.be/769Hmeq08OU

Solution

Let $x_i = \tan(a_i - \frac{\pi}{4})$. Then we have that

$$\tan a_i = \tan(a_i - 45^\circ + 45^\circ) = \frac{x_i + 1}{1 - x_i}.$$

If we further substitute $y_i = \frac{1-x_i}{2} \in (0,1)$, then we have to prove that the following statement:

Claim. If $\sum_0^n y_i \le 1$ and $y_i \ge 0$, we have

$$\prod_{i=1}^{n} \left(\frac{1}{y_i} - 1 \right) \ge n^{n+1}.$$

Proof. Homogenizing, we have to prove that

$$\prod_{i=1}^{n} \left(\frac{y_0 + y_1 + y_2 + \cdots + y_n}{y_i} - 1 \right) \ge n^{n+1}.$$

By AM-GM, we have

$$\frac{y_1 + y_2 + y_3 + \cdots + y_n}{y_0} \ge n \sqrt[n]{\frac{y_1 y_2 y_3 \cdots y_n}{y_1}}.$$

Cyclic product works. □

Remark. Alternatively, the function $x \mapsto \log(1/x - 1)$ is a convex function on $(0,1)$ so Jensen inequality should also work.