TSTST 2013/4

Evan Chen

Twitch Solves ISL

Episode 40

Problem

Circle ω, centered at X, is internally tangent to circle Ω, centered at Y, at T. Let P and S be variable points on Ω and ω, respectively, such that line PS is tangent to ω (at S). Determine the locus of O – the circumcenter of triangle PST.

Video

https://youtu.be/gUb0y5VD6As

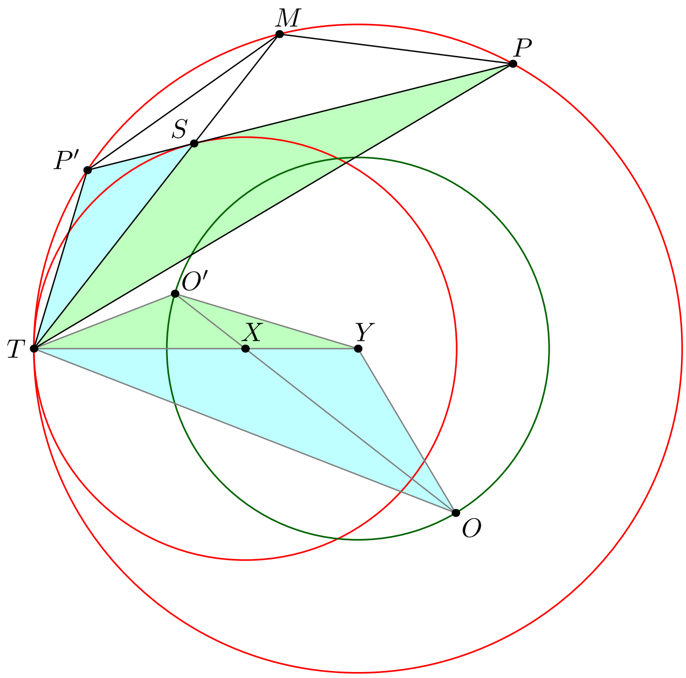
External Link

https://aops.com/community/p3181482

Solution

The answer is a circle centered at Y with radius $\sqrt{YX \cdot YT}$, minus the two points on line XY itself.

We let PS meet Ω again at P', and let O' be the circumcenter of $\triangle TPS'$. Note that O', X, O are collinear on the perpendicular bisector of line $\overline{TS}$ Finally, we let M denote the arc midpoint of PP' which lies on line TS (by homothety).



By three applications of Salmon theorem, we have the following spiral similarities all centered at T:

$$\triangle TSP \overset{+}{\sim} \triangle TO'Y$$
$$\triangle TP'S \overset{+}{\sim} \triangle TYO$$
$$\triangle TP'P \overset{+}{\sim} \triangle TO'O.$$

However, the shooting lemma also gives us two similarities:

$$\triangle TP'M \overset{+}{\sim} \triangle TSP$$
$$\triangle TMP \overset{+}{\sim} \triangle TP'S.$$

Putting everything together, we find that

$$TP'MP \overset{+}{\sim} TO'YO.$$

Then by shooting lemma, $YO'^2 = YX \cdot YT$, so O indeed lies on the claimed circle.

As the line $\overline{O'O}$ may be any line through X other than line XY (one takes S to be the reflection of T across this line) one concludes the only two non-achievable points are the diametrically opposite ones on line XY of this circle (because this leads to the only degenerate situation where $S = T$).