# TSTST 2020/3 <br> Evan Chen <br> Twitch Solves ISL <br> Episode 39 

## Problem

We say a nondegenerate triangle whose angles have measures $\theta_{1}, \theta_{2}, \theta_{3}$ is quirky if there exists integers $r_{1}, r_{2}, r_{3}$, not all zero, such that

$$
r_{1} \theta_{1}+r_{2} \theta_{2}+r_{3} \theta_{3}=0
$$

Find all integers $n \geq 3$ for which a triangle with side lengths $n-1, n, n+1$ is quirky.

## Video

https://youtu.be/zkygeEZ_scc

## External Link

https://aops.com/community/p18933954

## Solution

The answer is $n=3,4,5,7$.
We first introduce a variant of the $k$ th Chebyshev polynomials in the following lemma (which is standard, and easily shown by induction).

Lemma. For each $k \geq 0$ there exists $P_{k}(X) \in \mathbb{Z}[X]$, monic for $k \geq 1$ and with degree $k$, such that

$$
P_{k}\left(X+X^{-1}\right) \equiv X^{k}+X^{-k} .
$$

The first few are $P_{0}(X) \equiv 2, P_{1}(X) \equiv X, P_{2}(X) \equiv X^{2}-2, P_{3}(X) \equiv X^{3}-3 X$.
Suppose the angles of the triangle are $\alpha<\beta<\gamma$, so the law of cosines implies that

$$
2 \cos \alpha=\frac{n+4}{n+1} \quad \text { and } \quad 2 \cos \gamma=\frac{n-4}{n-1} .
$$

Claim. The triangle is quirky iff there exists $r, s \in \mathbb{Z}_{\geq 0}$ not both zero such that

$$
\cos (r \alpha)= \pm \cos (s \gamma) \quad \text { or equivalently } \quad P_{r}\left(\frac{n+4}{n+1}\right)= \pm P_{s}\left(\frac{n-4}{n-1}\right)
$$

Proof. If there are integers $x, y, z$ for which $x \alpha+y \beta+z \gamma=0$, then we have that $(x-y) \alpha=(y-z) \gamma-\pi y$, whence it follows that we may take $r=|x-y|$ and $s=|y-z|$ (noting $r=s=0$ implies the absurd $x=y=z$ ). Conversely, given such $r$ and $s$ with $\cos (r \alpha)= \pm \cos (s \gamma)$, then it follows that $r \alpha \pm s \gamma=k \pi=k(\alpha+\beta+\gamma)$ for some $k$, so the triangle is quirky.

If $r=0$, then by rational root theorem on $P_{s}(X) \pm 2$ it follows $\frac{n-4}{n-1}$ must be an integer which occurs only when $n=4$ (recall $n \geq 3$ ). Similarly we may discard the case $s=0$.

Thus in what follows assume $n \neq 4$ and $r, s>0$. Then, from the fact that $P_{r}$ and $P_{s}$ are nonconstant monic polynomials, we find
Corollary. If $n \neq 4$ works, then when $\frac{n+4}{n+1}$ and $\frac{n-4}{n-1}$ are written as fractions in lowest terms, the denominators have the same set of prime factors.

But $\operatorname{gcd}(n+1, n-1)$ divides 2 , and $\operatorname{gcd}(n+4, n+1), \operatorname{gcd}(n-4, n-1)$ divide 3. So we only have three possibilities:

- $n+1=2^{u}$ and $n-1=2^{v}$ for some $u, v \geq 0$. This is only possible if $n=3$. Here $2 \cos \alpha=\frac{7}{4}$ and $2 \cos \gamma=-\frac{1}{2}$, and indeed $P_{2}(-1 / 2)=-7 / 4$.
- $n+1=3 \cdot 2^{u}$ and $n-1=2^{v}$ for some $u, v \geq 0$, which implies $n=5$. Here $2 \cos \alpha=\frac{3}{2}$ and $2 \cos \gamma=\frac{1}{4}$, and indeed $P_{2}(3 / 2)=1 / 4$.
- $n+1=2^{u}$ and $n-1=3 \cdot 2^{v}$ for some $u, v \geq 0$, which implies $n=7$. Here $2 \cos \alpha=\frac{11}{8}$ and $2 \cos \gamma=\frac{1}{2}$, and indeed $P_{3}(1 / 2)=-11 / 8$.

Finally, $n=4$ works because the triangle is right, completing the solution.
Remark (Major generalization due to Luke Robitaille). In fact one may find all quirky triangles whose sides are integers in arithmetic progression.
Indeed, if the side lengths of the triangle are $x-y, x, x+y$ with $\operatorname{gcd}(x, y)=1$ then the problem becomes

$$
P_{r}\left(\frac{x+4 y}{x+y}\right)= \pm P_{s}\left(\frac{x-4 y}{x-y}\right)
$$

and so in the same way as before, we ought to have $x+y$ and $x-y$ are both of the form $3 \cdot 2^{*}$ unless $r s=0$. This time, when $r s=0$, we get the extra solutions $(1,0)$ and $(5,2)$.

For $r s \neq 0$, by triangle inequality, we have $x-y \leq x+y<3(x-y)$, and $\min \left(\nu_{2}(x-\right.$ $\left.y), \nu_{2}(x+y)\right) \leq 1$, so it follows one of $x-y$ or $x+y$ must be in $\{1,2,3,6\}$. An exhaustive check then leads to

$$
(x, y) \in\{(3,1),(5,1),(7,1),(11,5)\} \cup\{(1,0),(5,2),(4,1)\}
$$

as the solution set. And in fact they all work.
In conclusion the equilateral triangle, $3-5-7$ triangle (which has a $120^{\circ}$ angle) and $6-11-16$ triangle (which satisfies $B=3 A+4 C$ ) are exactly the new quirky triangles (up to similarity) whose sides are integers in arithmetic progression.

