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Evan Chen

Twitch Solves ISL

Episode 39

Problem

We say a nondegenerate triangle whose angles have measures θ_1, θ_2, θ_3 is *quirky* if there exists integers r_1, r_2, r_3, not all zero, such that

$$r_1\theta_1 + r_2\theta_2 + r_3\theta_3 = 0.$$

Find all integers $n \geq 3$ for which a triangle with side lengths $n-1$, n, $n+1$ is quirky.

Video

https://youtu.be/ePkZIVu9YJs

Solution

The answer is $n = 3,\ 4,\ 5,\ 7$.

We first introduce a variant of the kth Chebyshev polynomials in the following lemma (which is standard, and easily shown by induction).

Lemma. For each $k \ge 0$ there exists $P_k(X) \in \mathbb{Z}[X]$, monic for $k \ge 1$ and with degree k, such that

$$P_k(X + X^{-1}) \equiv X^k + X^{-k}.$$

The first few are $P_0(X) \equiv 2$, $P_1(X) \equiv X$, $P_2(X) \equiv X^2 - 2$, $P_3(X) \equiv X^3 - 3X$.

Suppose the angles of the triangle are $\alpha < \beta < \gamma$, so the law of cosines implies that

$$2\cos\alpha = \frac{n+4}{n+1} \qquad \text{and} \qquad 2\cos\gamma = \frac{n-4}{n-1}.$$

Claim. The triangle is quirky iff there exists $r, s \in \mathbb{Z}_{\ge 0}$ not both zero such that

$$\cos(r\alpha) = \pm\cos(s\gamma) \qquad \text{or equivalently} \qquad P_r\left(\frac{n+4}{n+1}\right) = \pm P_s\left(\frac{n-4}{n-1}\right).$$

Proof. If there are integers x, y, z for which $x\alpha + y\beta + z\gamma = 0$, then we have that $(x-y)\alpha = (y-z)\gamma - \pi y$, whence it follows that we may take $r = |x-y|$ and $s = |y-z|$ (noting $r = s = 0$ implies the absurd $x = y = z$). Conversely, given such r and s with $\cos(r\alpha) = \pm\cos(s\gamma)$, then it follows that $r\alpha \pm s\gamma = k\pi = k(\alpha+\beta+\gamma)$ for some k, so the triangle is quirky. $\square$

If $r = 0$, then by rational root theorem on $P_s(X) \pm 2$ it follows $\frac{n-4}{n-1}$ must be an integer which occurs only when $n = 4$ (recall $n \ge 3$). Similarly we may discard the case $s = 0$.

Thus in what follows assume $n \ne 4$ and $r, s > 0$. Then, from the fact that P_r and P_s are nonconstant monic polynomials, we find

Corollary. If $n \ne 4$ works, then when $\frac{n+4}{n+1}$ and $\frac{n-4}{n-1}$ are written as fractions in lowest terms, the denominators have the same set of prime factors.

But $\gcd(n+1, n-1)$ divides 2, and $\gcd(n+4, n+1)$, $\gcd(n-4, n-1)$ divide 3. So we only have three possibilities:

- $n + 1 = 2^u$ and $n - 1 = 2^v$ for some $u, v \ge 0$. This is only possible if $n = 3$. Here $2\cos\alpha = \frac{7}{4}$ and $2\cos\gamma = -\frac{1}{2}$, and indeed $P_2(-1/2) = -7/4$.
- $n + 1 = 3 \cdot 2^u$ and $n - 1 = 2^v$ for some $u, v \ge 0$, which implies $n = 5$. Here $2\cos\alpha = \frac{3}{2}$ and $2\cos\gamma = \frac{1}{4}$, and indeed $P_2(3/2) = 1/4$.
- $n + 1 = 2^u$ and $n - 1 = 3 \cdot 2^v$ for some $u, v \ge 0$, which implies $n = 7$. Here $2\cos\alpha = \frac{11}{8}$ and $2\cos\gamma = \frac{1}{2}$, and indeed $P_3(1/2) = -11/8$.

Finally, $n = 4$ works because the triangle is right, completing the solution.

Remark (Major generalization due to Luke Robitaille)**.** In fact one may find all quirky triangles whose sides are integers in arithmetic progression.

Indeed, if the side lengths of the triangle are $x - y$, x, $x + y$ with $\gcd(x, y) = 1$ then the problem becomes

$$P_r\left(\frac{x+4y}{x+y}\right) = \pm P_s\left(\frac{x-4y}{x-y}\right)$$

and so in the same way as before, we ought to have $x+y$ and $x-y$ are both of the form $3 \cdot 2^*$ unless $rs = 0$. This time, when $rs = 0$, we get the extra solutions $(1,0)$ and $(5,2)$.

For $rs \neq 0$, by triangle inequality, we have $x - y \le x + y < 3(x-y)$, and $\min(\nu_2(x-y), \nu_2(x+y)) \le 1$, so it follows one of $x-y$ or $x+y$ must be in $\{1,2,3,6\}$. An exhaustive check then leads to

$$(x,y) \in \{(3,1),(5,1),(7,1),(11,5)\} \cup \{(1,0),(5,2),(4,1)\}$$

as the solution set. And in fact they all work.

In conclusion the equilateral triangle, $3-5-7$ triangle (which has a 120° angle) and $6-11-16$ triangle (which satisfies $B = 3A + 4C$) are exactly the new quirky triangles (up to similarity) whose sides are integers in arithmetic progression.