TSTST 2020/2 Evan Chen

Twitch Solves ISL

Episode 39

Problem

Let ABC be a scalene triangle with incenter I. The incircle of ABC touches \overline{BC} , \overline{CA} , \overline{AB} at points D, E, F, respectively. Let P be the foot of the altitude from D to \overline{EF} , and let M be the midpoint of \overline{BC} . The rays AP and IP intersect the circumcircle of triangle ABC again at points G and Q, respectively. Show that the incenter of triangle GQM coincides with D.

Video

https://youtu.be/zkygeEZ_scc

External Link

https://aops.com/community/p18933557

Solution

Refer to the figure below.



Claim. The point Q is the Miquel point of BFEC. Also, \overline{QD} bisects $\angle BQC$.

Proof. Inversion around the incircle maps line EF to (AIEF) and the nine-point circle of $\triangle DEF$ to the circumcircle of $\triangle ABC$ (as the midpoint of \overline{EF} maps to A, etc.). This implies P maps to Q; that is, Q coincides with the second intersection of (AFIE) with (ABC). This is the claimed Miquel point.

The spiral similarity mentioned then gives $\frac{QB}{BF} = \frac{QC}{CE}$, so \overline{QD} bisects $\angle BQC$.

Remark. The point Q and its properties mentioned in the first claim have appeared in other references. See for example Canada 2007/5, ELMO 2010/6, HMMT 2016 T-10, USA TST 2017/2, USA TST 2019/6 for a few examples.

Claim. We have (QG; BC) = -1, so in particular \overline{GD} bisects $\angle BGC$.

Proof. Note that

$$-1 = (AI; EF) \stackrel{Q}{=} (\overline{AQ} \cap \overline{EF}, P; E, F) \stackrel{A}{=} (QG; BC).$$

The last statement follows from Apollonian circle, or more bluntly $\frac{GB}{GC} = \frac{QB}{QC} = \frac{BD}{DC}$. \Box

Hence \overline{QD} and \overline{GD} are angle bisectors of $\angle BQC$ and $\angle BGC$. However, \overline{QM} and \overline{QG} are isogonal in $\angle BQC$ (as median and symmetrian), and similarly for $\angle BGC$, as desired.