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Twitch Solves ISL

Episode 39

Problem

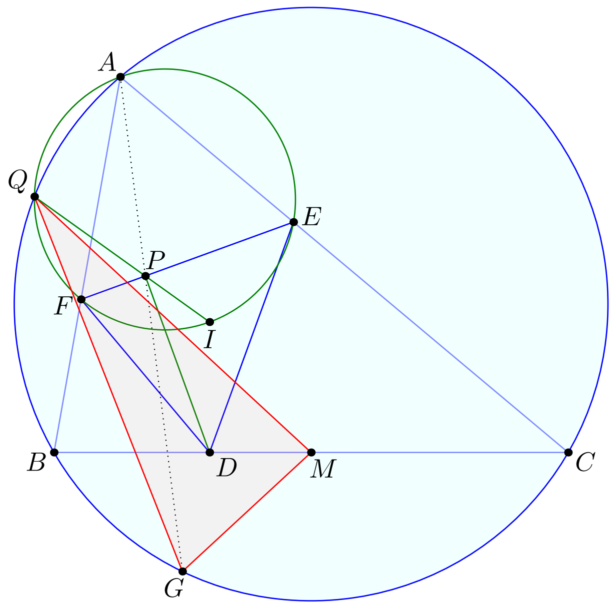
Let ABC be a scalene triangle with incenter I. The incircle of ABC touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at points D, E, F, respectively. Let P be the foot of the altitude from D to $\overline{EF}$, and let M be the midpoint of $\overline{BC}$. The rays AP and IP intersect the circumcircle of triangle ABC again at points G and Q, respectively. Show that the incenter of triangle GQM coincides with D.

Video

https://youtu.be/ePkZIVu9YJs

Solution

Refer to the figure below.



Claim. The point Q is the Miquel point of $BFEC$. Also, $\overline{QD}$ bisects $\angle BQC$.

Proof. Inversion around the incircle maps line EF to $(AIEF)$ and the nine-point circle of $\triangle DEF$ to the circumcircle of $\triangle ABC$ (as the midpoint of $\overline{EF}$ maps to A, etc.). This implies P maps to Q; that is, Q coincides with the second intersection of $(AFIE)$ with (ABC). This is the claimed Miquel point.

The spiral similarity mentioned then gives $\frac{QB}{BF} = \frac{QC}{CE}$, so $\overline{QD}$ bisects $\angle BQC$. $\square$

Remark. The point Q and its properties mentioned in the first claim have appeared in other references. See for example Canada 2007/5, ELMO 2010/6, HMMT 2016 T-10, USA TST 2017/2, USA TST 2019/6 for a few examples.

Claim. We have $(QG; BC) = -1$, so in particular $\overline{GD}$ bisects $\angle BGC$.

Proof. Note that

$$-1 = (AI; EF) \overset{Q}{=} (\overline{AQ} \cap \overline{EF}, P; E, F) \overset{A}{=} (QG; BC).$$

The last statement follows from Apollonian circle, or more bluntly $\frac{GB}{GC} = \frac{QB}{QC} = \frac{BD}{DC}$. $\square$

Hence $\overline{QD}$ and $\overline{GD}$ are angle bisectors of $\angle BQC$ and $\angle BGC$. However, $\overline{QM}$ and $\overline{QG}$ are isogonal in $\angle BQC$ (as median and symmedian), and similarly for $\angle BGC$, as desired.