TSTST 2020/1

Evan Chen

Twitch Solves ISL

Episode 39

Problem

Let a, b, c be fixed positive integers. There are $a+b+c$ ducks sitting in a circle, one behind the other. Each duck picks either *rock*, *paper*, or *scissors*, with a ducks picking rock, b ducks picking paper, and c ducks picking scissors.

A *move* consists of an operation of one of the following three forms:

- If a duck picking rock sits behind a duck picking scissors, they switch places.
- If a duck picking paper sits behind a duck picking rock, they switch places.
- If a duck picking scissors sits behind a duck picking paper, they switch places.

Determine, in terms of a, b, and c, the maximum number of moves which could take place, over all possible initial configurations.

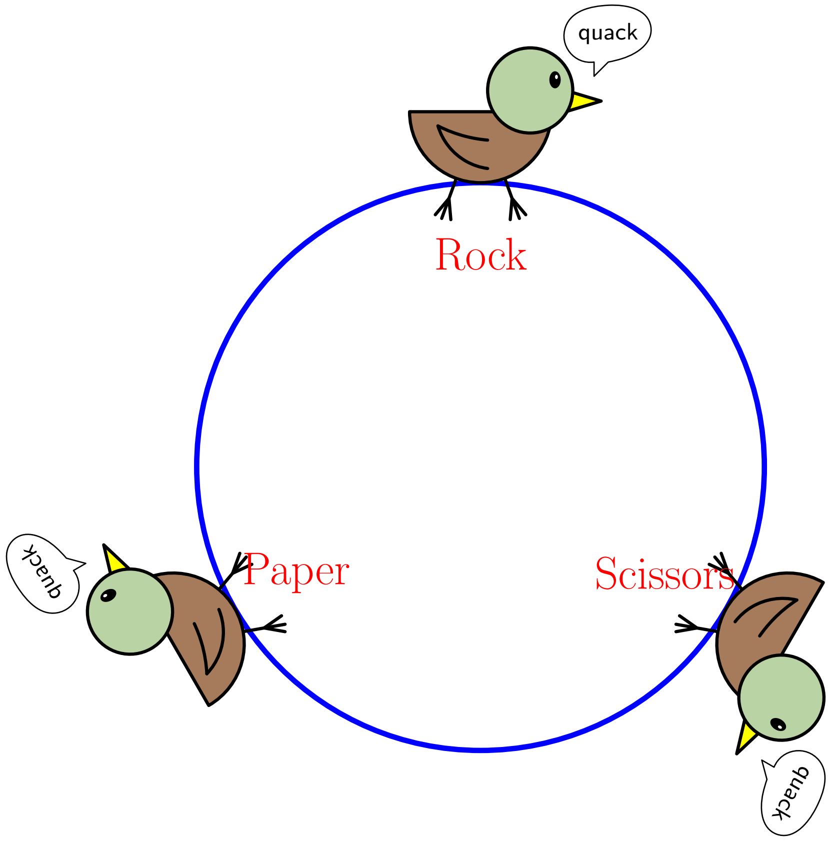
Video

https://youtu.be/ePkZIVu9YJs

Solution

The maximum possible number of moves is $\max(ab, ac, bc)$.

First, we prove this is best possible. We define a *feisty triplet* to be an unordered triple of ducks, one of each of rock, paper, scissors, such that the paper duck is between the rock and scissors duck and facing the rock duck, as shown. (There may be other ducks not pictured, but the orders are irrelevant.)

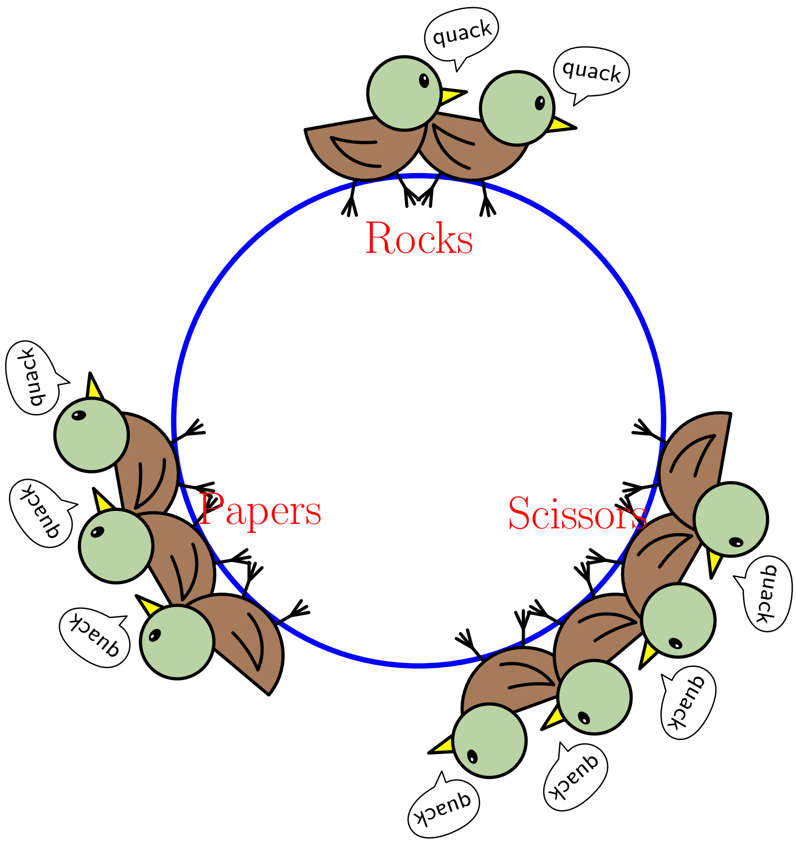


Claim. The number of feisty triplets decreases by c if a paper duck swaps places with a rock duck, and so on.

Proof. Clear. □

Obviously the number of feisty triples is at most abc to start. Thus at most $\max(ab, bc, ca)$ moves may occur, since the number of feisty triplets should always be nonnegative, at which point no moves are possible at all.

To see that this many moves is possible, assume WLOG $a = \min(a, b, c)$ and suppose we have a rocks, b papers, and c scissors in that clockwise order.



Then, allow the scissors to filter through the papers while the rocks stay put. Each of the b papers swaps with c scissors, for a total of $bc = \max(ab, ac, bc)$ swaps.

Remark (Common errors)**.** One small possible mistake: it is not quite kösher to say that "WLOG $a \le b \le c$" because the condition is not symmetric, only cyclic. Therefore in this solution we only assume $a = \min(a, b, c)$.

It is true here that every pair of ducks swaps at most once, and some solutions make use of this fact. However, this fact implicitly uses the fact that $a, b, c > 0$ and is false without this hypothesis.