Shortlist 1995 N8

Evan Chen

Twitch Solves ISL

Episode 38

Problem

Let p be an odd prime. Determine positive integers x and y for which $x \le y$ and $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is non-negative and as small as possible.

Video

https://youtu.be/ePkZIVu9YJs

External Link

https://aops.com/community/p1219477

Solution

We claim the best case is if $x = \frac{p-1}{2}$ and $y = \frac{p+1}{2}$. The fact that $\sqrt{x} + \sqrt{y} \le \sqrt{2p}$ follows by Jensen. Moreover, it is best possible among choices with $x + y = p$ by Karamata, and hence is also better than any choice with $x + y < p$ (since (x, y) will be worse than $(x, p - y)$).

We are left to show that we cannot have

$$\sqrt{\frac{p-1}{2}} + \sqrt{\frac{p+1}{2}} < \sqrt{x} + \sqrt{y} < \sqrt{2p}$$

for positive integers x and y satisfying $x + y > p$. Let $z = x + y$ and $t = x - y$.

Claim (Calculation)**.** Let $A = z + \sqrt{z^2 - t^2}$. Then we have $A < 2p < A + \frac{1}{A}$.

Proof. Square both sides:

$$\begin{aligned} \sqrt{\frac{p-1}{2}} + \sqrt{\frac{p+1}{2}} &< \sqrt{x} + \sqrt{y} < \sqrt{2p} \\ \sqrt{p^2-1} + p &< x + y + 2\sqrt{xy} < 2p \\ \sqrt{p^2-1} &< x + y - p + 2\sqrt{xy} < p. \end{aligned}$$

which simplifies to

$$p^2 - 1 < (x + y - p)^2 + 4xy + 4(x + y - p)\sqrt{xy} < p^2.$$

Let $z = x + y$ and $t = x - y$. Simplify further:

$$\begin{aligned} & p^2 - 1 < (z-p)^2 + (z^2 - t^2) + 2(z-p)\sqrt{z^2-t^2} < p^2 \\ \iff & t^2 - 1 < 2(z-p)\left[z + \sqrt{z^2-t^2}\right] < t^2 \\ \iff & (1 - t^{-2})(z - \sqrt{z^2-t^2}) < 2(z-p) < z - \sqrt{z^2-t^2} \\ \iff & z + \sqrt{z^2-t^2} < 2p < z + \sqrt{z^2-t^2} + \frac{1}{z + \sqrt{z^2-t^2}}. \end{aligned}$$

Hence done. □

The idea from here is that "square roots don't approximate integers too well". We formalize this using the following lemma.

Lemma. Let $n \ge 2$ be an integer which is not a perfect square. Then $n - \sqrt{n^2-1} > \frac{1}{2n}$.

Proof. Write $n - \sqrt{n^2-1} = \frac{1}{n + \sqrt{n^2-1}}$. □

Assume for contradiction A really exists with

$$A < 2p < A + \frac{1}{A}$$

and $z > p$. If A is an integer, we have a contradiction. Else, since $z \ge p + 1$ it follows that the fractional part of A satisfies

$$1 - \frac{1}{2(2p-z)} \overset{\text{lemma}}{\ge} \{\sqrt{z^2-t^2}\} = \{A\} > 1 - \frac{1}{A}$$

hence

$$A < 2(2p - z) \le 2\left[2p - (p+1)\right] = 2p - 2$$

which is a contradiction.