RMM 2016/5

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Twitch Solves ISL

Episode 38

Problem

A convex hexagon $A_1B_1A_2B_2A_3B_3$ is inscribed in a circle Ω with radius R. The diagonals A_1B_2, A_2B_3, A_3B_1 are concurrent in X. For each $i = 1, 2, 3$ let ω_i tangent to the segments XA_i and XB_i and tangent to the arc A_iB_i of Ω that does not contain the other vertices of the hexagon; let r_i the radius of ω_i.

(a) Prove that $R \geq r_1 + r_2 + r_3$.

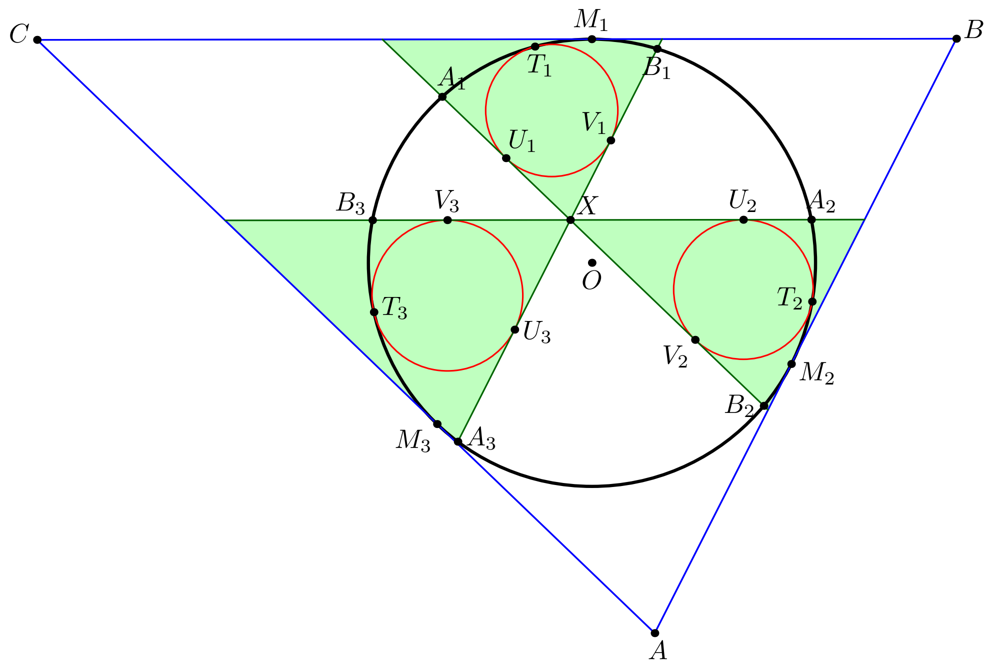
(b) If $R = r_1 + r_2 + r_3$, prove that the six points of tangency of the circumferences ω_i with the diagonals A_1B_2, A_2B_3, A_3B_1 are concyclic.

Video

https://youtu.be/ZqJNTYGwl8A

Solution

We let M_1 be the arc midpoint of the arc A_2B_3 containing A_1B_1, and so on. The tangents to Ω at M_1, M_2, M_3 determine a triangle ABC with incircle Ω. We also let ω_i be tangent to Ω at T_i, and let U_i and V_i be tangency points as in the figure below.



The three green segments drawn (the extension of A_1B_2, A_2B_3, A_3B_1) together with the sides of triangle ABC determine three smaller triangles, shaded light green in the diagram above. Notice that

- The red circles have radius smaller than the inradius of the green triangles they are contained in
- The sum of the three inradii of the green triangles equals R, because the homothety factors between the green triangles sum to 1.

This immediately implies that $R \geq r_1 + r_2 + r_3$, so (a) is solved. Moreover, equality holds if and only if $M_i = T_i$ for each i.

To finish (b), note that T_3M_1 should unconditionally pass through the tangency point of ω_1 with $\overline{A_1B_2}$, etc. Hence we always have

$$M_3M_1 \cdot M_3U_1 = M_3V_2 \cdot M_3M_2.$$

However, we also have

$$\frac{T_3V_3}{T_3M_1} = \frac{T_3U_3}{T_3M_2}.$$

So if $M_i = T_i$ for all i, then by power of a point from $T_3 = M_3$ we are able to conclude that U_1, V_3, U_3, V_2 are cyclic.

Analogous statements imply that unless U_1V_1, U_2V_2, U_3V_3 are cyclic — which they obviously are not, because they are perpendicular to angle bisectors of $\angle A_iXB_i$ — then the six points must be cyclic, as desired.