

USAMO 1999/3

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TWITCH SOLVES ISL

Episode 37

Problem

Let $p > 2$ be a prime and let a, b, c, d be integers not divisible by p , such that

$$\left\{ \frac{ra}{p} \right\} + \left\{ \frac{rb}{p} \right\} + \left\{ \frac{rc}{p} \right\} + \left\{ \frac{rd}{p} \right\} = 2$$

for any integer r not divisible by p . (Here, $\{t\} = t - \lfloor t \rfloor$ is the fractional part.) Prove that at least two of the numbers $a + b, a + c, a + d, b + c, b + d, c + d$ are divisible by p .

Video

<https://youtu.be/LNBcuBMmN9g>

Solution

First of all, we apparently have $r(a + b + c + d) \equiv 0 \pmod{p}$ for every prime p , so it automatically follows that $a + b + c + d \equiv 0 \pmod{p}$. By scaling appropriately, and also replacing each number with its remainder modulo p , we are going to assume that

$$1 = a \leq b \leq c \leq d < p.$$

We are going to prove that $d = p - 1$, which will solve the problem.

Claim. For each integer $r = 1, 2, \dots, p - 1$ we have

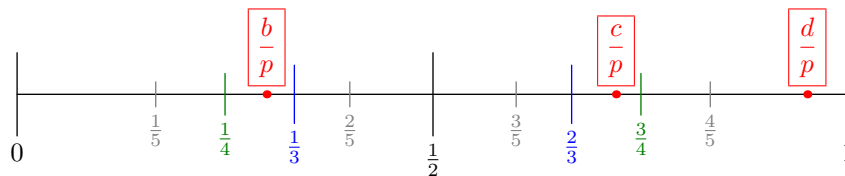
$$2(r - 1) = \left\lfloor \frac{rb}{p} \right\rfloor + \left\lfloor \frac{rc}{p} \right\rfloor + \left\lfloor \frac{rd}{p} \right\rfloor.$$

Proof. By plugging in $r = 1$ to the given we have $a + b + c + d = 2p$. Now, we have

$$2 = \sum_{\text{cyc}} \left(\frac{ra}{p} - \left\lfloor \frac{ra}{p} \right\rfloor \right)$$

and since $a + b + c + d = 2p$ the conclusion follows. □

We vaguely outline the approach now, before giving a formalization. Imagine the interval $[0, 1]$. One by one, for each $r = 1, 2, 3, \dots, p - 1$, we mark the fractions with denominator r on this number line; the resulting pictures may be better known as *Farey fractions*. At each step, we can place the three numbers $b/p, c/p, d/p$ into one of the resulting sub-intervals. Our goal is to show that d/p is always in the rightmost interval, while b/p and c/p are always to the right of symmetrically mirrored points. An example of a possible diagram is shown below (not to scale).



In symbols, It will be enough to prove the following.

Claim. For each $r = 1, 2, \dots, p - 2$ we have $\frac{r-1}{r} < \frac{d}{p} < 1$.

Equivalently, for each $r = 1, 2, \dots, p - 2$ we have $\left\lfloor \frac{rb}{p} \right\rfloor + \left\lfloor \frac{rc}{p} \right\rfloor = r - 1$.

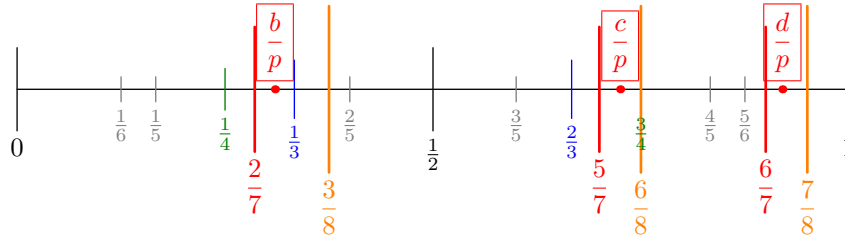
Proof. Assume this is not true and take the minimal counterexample $r > 1$. Then evidently

$$r - 1 > \left\lfloor \frac{rd}{p} \right\rfloor \geq \left\lfloor \frac{(r - 1)d}{p} \right\rfloor = r - 2.$$

Now, we have that

$$2(r - 1) = \left\lfloor \frac{rb}{p} \right\rfloor + \left\lfloor \frac{rc}{p} \right\rfloor + \underbrace{\left\lfloor \frac{rd}{p} \right\rfloor}_{=r-2}.$$

Thus $\left\lfloor \frac{rb}{p} \right\rfloor > \left\lfloor \frac{(r-1)b}{p} \right\rfloor$, and $\left\lfloor \frac{rc}{p} \right\rfloor > \left\lfloor \frac{(r-1)c}{p} \right\rfloor$. An example of this situation is illustrated below with $r = 7$ (not to scale).



Right now, $\frac{b}{p}$ and $\frac{c}{p}$ are just to the right of $\frac{u}{r}$ and $\frac{v}{r}$ for some u and v with $u + v = r$. The issue is that there is some fraction just to the right of $\frac{b}{p}$ and $\frac{c}{p}$ from an earlier value of r , and by hypothesis its denominator is going to be strictly greater than 1.

It is at this point we are going to use the properties of Farey sequences. When we consider the fractions with denominator $r + 1$, they are going to lie outside of the interval they we have constrained $\frac{b}{p}$ and $\frac{c}{p}$ to lie in. Indeed, if right now $\frac{u}{r} < \frac{b}{p} < \frac{s}{t}$, then Farey theory says the next fraction inside the interval $[\frac{u}{v}, \frac{s}{t}]$ is $\frac{u+s}{v+t}$, and since $t > 1$, we have $v + t > r + 1$. In other words, we get an inequality of the form

$$\frac{u}{r} < \frac{b}{p} < \text{something} \leq \frac{u + 1}{r + 1}.$$

The same holds for $\frac{c}{p}$ as

$$\frac{v}{r} < \frac{c}{p} < \text{something} \leq \frac{v + 1}{r + 1}.$$

Finally,

$$\frac{d}{p} < \frac{r - 1}{r} < \frac{r}{r + 1}.$$

So now we have that

$$\left\lfloor \frac{(r + 1)b}{p} \right\rfloor + \left\lfloor \frac{(r + 1)c}{p} \right\rfloor + \left\lfloor \frac{(r + 1)d}{p} \right\rfloor \leq u + v + (r - 1) = 2r - 1$$

which is a contradiction. □

Now, since

$$\frac{p - 3}{p - 2} < \frac{d}{p} \implies d > \frac{p(p - 3)}{p - 2} = p - 1 - \frac{2}{p - 2}$$

which for $p > 2$ gives $d = p - 1$.