

Twitch 037.3

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TWITCH SOLVES ISL

Episode 37

Problem

If $x + y = 2$ for real numbers $x, y > 0$, prove that $\sqrt{x^2 + 3} + \sqrt{y^2 + 3} + \sqrt{xy + 3} \geq 6$.

Video

<https://youtu.be/Rsr1cKKDljs>

Solution

Square both sides and rewrite as

$$\begin{aligned}
 & \sqrt{x^2 + 3} + \sqrt{y^2 + 3} \geq 6 - \sqrt{xy + 3} \\
 \iff & x^2 + y^2 + 6 + 2\sqrt{(x^2 + 3)(y^2 + 3)} \geq 39 + xy - 12\sqrt{xy + 3} \\
 \iff & x^2 + y^2 + 12\sqrt{xy + 3} + 2\sqrt{(x^2 + 3)(y^2 + 3)} \geq 33 + xy \\
 \iff & 12\sqrt{xy + 3} + 2\sqrt{(x^2 + 3)(y^2 + 3)} \geq 29 + 3xy \\
 \iff & 12\sqrt{xy + 3} + 2\sqrt{x^2y^2 - 6xy + 21} \geq 29 + 3xy.
 \end{aligned}$$

Let $t = xy$, so want $12\sqrt{t + 3} + 2\sqrt{t^2 - 6t + 21} \geq 29 + 3t$. Our constraint on t is $t \leq 1$. Squaring both sides, this is

$$144(t + 3) + 4(t^2 - 6t + 21) + 24\sqrt{(t^2 - 6t + 21)(t + 3)} \geq 9t^2 + 174t + 841$$

which is equivalent to

$$48\sqrt{(t^2 - 6t + 21)(t + 3)} \geq 5t^2 + 54t + 325.$$

Squaring both sides again, we find this is equivalent to

$$\begin{aligned}
 & 2304t^3 - 6912t^2 + 6912t + 145152 \\
 & \geq 25t^4 + 540t^3 + 6166t^2 + 35100t + 105625 \\
 \iff & 0 \geq 25t^4 - 1764t^3 + 13078t^2 + 28188t - 39527 \\
 & = (t - 1)(25t^3 - 1739t^2 + 11339t + 39527)
 \end{aligned}$$

which is obviously true for $t \leq 1$.