

Shortlist 2003 N8

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TWITCH SOLVES ISL

Episode 37

Problem

Let p be a prime number and let A be a set of positive integers that satisfies the following conditions:

- (i) the set of prime divisors of the elements in A consists of $p - 1$ elements;
- (ii) for any nonempty subset of A , the product of its elements is not a perfect p -th power.

What is the largest possible number of elements in A ?

Video

<https://youtu.be/0Ax32Q0Z0Ho>

External Link

<https://aops.com/community/p119994>

Solution

Let $D = p - 1$. Then the question (thinking in terms of the exponents) can be phrased as follows:

What's the largest multiset of vectors in \mathbb{F}_p^D such that no nonempty subset has zero sum?

We claim the answer is $D \cdot (p - 1)$. A construction is given by taking

- $p - 1$ copies of the vector $\langle 1, 0, \dots, 0 \rangle$;
- $p - 1$ copies of the vector $\langle 0, 1, \dots, 0 \rangle$;
- \dots ;
- $p - 1$ copies of the vector $\langle 0, 0, \dots, 1 \rangle$.

To show it's best possible, suppose we have vectors v_1, \dots, v_N with coordinates given as

$$v_k = \langle v_{k,1}, v_{k,2}, \dots, v_{k,D} \rangle \quad k = 1, 2, \dots, N.$$

Then, we construct the polynomial

$$F(X_1, \dots, X_N) = \prod_{i=1}^D \left[1 - \left(\sum_{k=1}^N X_k v_{k,i} \right)^{p-1} \right] - \prod_{i=1}^N (1 - X_i)$$

If we imagine the $X_i \in \{0, 1\}$ as Bernoulli random variables indicating whether v_k is used in a set or not, then $F(X_1, \dots, X_N) \neq 0$ exactly when the X_i 's equal to 1 correspond to a nonempty subset of the vectors which have vanishing sum.

Now assume for contradiction $N < D \cdot (p - 1)$. Then the largest degree term is given in the latter product; it is exactly $(-1)^{N+1} X_1 X_2 \dots X_N$. So if we quote Alon's combinatorial nullstellensatz, it now follows that there is a choice of $X_i \in \{0, 1\}$ such that $F(X_1, \dots, X_N) \neq 0$ which is a contradiction.