Shortlist 2003 N8

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Episode 37

Problem

Let p be a prime number and let A be a set of positive integers that satisfies the following conditions:

(i) the set of prime divisors of the elements in A consists of $p-1$ elements;

(ii) for any nonempty subset of A, the product of its elements is not a perfect p-th power.

What is the largest possible number of elements in A ?

Video

https://youtu.be/0Ax32Q0Z0Ho

Solution

Let $D = p - 1$. Then the question (thinking in terms of the exponents) can be phrased as follows:

> What's the largest multiset of vectors in $\mathbb{F}_p^D$ such that no nonempty subset has zero sum?

We claim the answer is $D \cdot (p-1)$. A construction is given by taking

- $p-1$ copies of the vector $\langle 1, 0, \dots, 0 \rangle$;
- $p-1$ copies of the vector $\langle 0, 1, \dots, 0 \rangle$;
- $\dots$;
- $p-1$ copies of the vector $\langle 0, 0, \dots, 1 \rangle$.

To show it's best possible, suppose we have vectors $v_1, \dots, v_N$ with coordinates given as

$$v_k = \langle v_{k,1}, v_{k,2}, \dots, v_{k,D} \rangle \qquad k = 1, 2, \dots, N.$$

Then, we construct the polynomial

$$F(X_1, \dots, X_N) = \prod_{i=1}^{D} \left[1 - \left(\sum_{k=1}^{N} X_k v_{k,j} \right)^{p-1} \right] - \prod_{i=1}^{N} (1 - X_i)$$

If we imagine the $X_i \in \{0, 1\}$ as Bernoulli random variables indicating whether v_k is used in a set or not, then $F(X_1, \dots, X_N) \neq 0$ exactly when the X_i's equal to 1 correspond to a nonempty subset of the vectors which have vanishing sum.

Now assume for contradiction $N < D \cdot (p-1)$. Then the largest degree term is given in the latter product; it is exactly $(-1)^{N+1} X_1 X_2 \dots X_N$. So if we quote Alon's combinatorial nullstellensatz, it now follows that there is a choice of $X_i \in \{0, 1\}$ such that $F(X_1, \dots, X_N) \neq 0$ which is a contradiction.