

# Shortlist 2003 N8

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TWITCH SOLVES ISL

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## Problem

Let  $p$  be a prime number and let  $A$  be a set of positive integers that satisfies the following conditions:

- (i) the set of prime divisors of the elements in  $A$  consists of  $p - 1$  elements;
- (ii) for any nonempty subset of  $A$ , the product of its elements is not a perfect  $p$ -th power.

What is the largest possible number of elements in  $A$  ?

## Video

<https://youtu.be/0Ax32Q0Z0Ho>

## External Link

<https://aops.com/community/p119994>

## Solution

Let  $D = p - 1$ . Then the question (thinking in terms of the exponents) can be phrased as follows:

What's the largest multiset of vectors in  $\mathbb{F}_p^D$  such that no nonempty subset has zero sum?

We claim the answer is  $D \cdot (p - 1)$ . A construction is given by taking

- $p - 1$  copies of the vector  $\langle 1, 0, \dots, 0 \rangle$ ;
- $p - 1$  copies of the vector  $\langle 0, 1, \dots, 0 \rangle$ ;
- ...;
- $p - 1$  copies of the vector  $\langle 0, 0, \dots, 1 \rangle$ .

To show it's best possible, suppose we have vectors  $v_1, \dots, v_N$  with coordinates given as

$$v_k = \langle v_{k,1}, v_{k,2}, \dots, v_{k,D} \rangle \quad k = 1, 2, \dots, N.$$

Then, we construct the polynomial

$$F(X_1, \dots, X_N) = \prod_{i=1}^D \left[ 1 - \left( \sum_{k=1}^N X_k v_{k,i} \right)^{p-1} \right] - \prod_{i=1}^N (1 - X_i)$$

If we imagine the  $X_i \in \{0, 1\}$  as Bernoulli random variables indicating whether  $v_k$  is used in a set or not, then  $F(X_1, \dots, X_N) \neq 0$  exactly when the  $X_i$ 's equal to 1 correspond to a nonempty subset of the vectors which have vanishing sum.

Now assume for contradiction  $N < D \cdot (p - 1)$ . Then the largest degree term is given in the latter product; it is exactly  $(-1)^{N+1} X_1 X_2 \dots X_N$ . So if we quote Alon's combinatorial nullstellensatz, it now follows that there is a choice of  $X_i \in \{0, 1\}$  such that  $F(X_1, \dots, X_N) \neq 0$  which is a contradiction.