# IMO 1982/1 

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## Twitch Solves ISL

Episode 37

## Problem

The function $f: \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{\geq 0}$ satisfies $f(2)=0, f(3)>0, f(9999)=3333$ and

$$
f(m+n)-f(m)-f(n) \in\{0,1\} \quad \text { for all } m, n>0
$$

Calculate $f(1982)$.

## Video

https://youtu.be/ls0fTee5eT0

## External Link

https://aops.com/community/p366626

## Solution

Answer: $f(1982)=660$ only.
We start with a few easy observations:

- Since $f(2)-2 f(1)=-2 f(1) \in\{0,1\}$ it follows $f(1)=0$.
- Since $f(3)-f(1)-f(2)=f(3) \in\{0,1\}$ it follows $f(3)=1$.

Now, we have the following general claim.
Claim. We must have $f(n+3) \geq f(n)+1$. Hence, by induction, $f(3 k) \geq k$ for every $k \geq 0$.

Proof.

$$
0 \leq f(n+3)-f(n)-f(3)=f(n+3)-(f(n)+1)
$$

However, since $f(9999)=3333$ exactly, it follows that equality must hold in the claim up until then. In other words, we have that $f(3 k)=k$ for each $k \geq 0$.

To finish, use the following calculation:

$$
\begin{aligned}
3333=f(9999) & \geq 3 f(1982)+f(4053)=3 f(1982)+1351 \\
\Rightarrow f(1982) & \leq \frac{1982}{3} \\
\text { but also } f(1982) & \geq f(1980)+f(2)=660+0=660
\end{aligned}
$$

This implies $f(1982)=660$.

