IMO 1982/1

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Twitch Solves ISL

Episode 37

Problem

The function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\ge 0}$ satisfies $f(2) = 0$, $f(3) > 0$, $f(9999) = 3333$ and

$$f(m+n) - f(m) - f(n) \in \{0, 1\} \qquad \text{for all } m, n > 0.$$

Calculate $f(1982)$.

Video

https://youtu.be/ls0fTee5eT0

Solution

Answer: $f(1982) = 660$ only.

We start with a few easy observations:

- Since $f(2) - 2f(1) = -2f(1) \in \{0, 1\}$ it follows $\boxed{f(1) = 0}$.
- Since $f(3) - f(1) - f(2) = f(3) \in \{0, 1\}$ it follows $\boxed{f(3) = 1}$.

Now, we have the following general claim.

Claim. We must have $f(n + 3) \geq f(n) + 1$. Hence, by induction, $f(3k) \geq k$ for every $k \geq 0$.

Proof.

$$0 \leq f(n + 3) - f(n) - f(3) = f(n + 3) - (f(n) + 1). \qquad \square$$

However, since $f(9999) = 3333$ exactly, it follows that equality must hold in the claim up until then. In other words, we have that $f(3k) = k$ for each $k \geq 0$.

To finish, use the following calculation:

$$\begin{aligned} 3333 = f(9999) &\geq 3f(1982) + f(4053) = 3f(1982) + 1351 \\ \implies f(1982) &\leq \frac{1982}{3} \\ \text{but also } f(1982) &\geq f(1980) + f(2) = 660 + 0 = 660. \end{aligned}$$

This implies $f(1982) = 660$.