IMO 1982/1 Evan Chen

TWITCH SOLVES ISL

Episode 37

Problem

The function $f: \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ satisfies f(2) = 0, f(3) > 0, f(9999) = 3333 and

 $f(m+n) - f(m) - f(n) \in \{0,1\}$ for all m, n > 0.

Calculate f(1982).

Video

https://youtu.be/ls0fTee5eT0

External Link

https://aops.com/community/p366626

Solution

Answer: f(1982) = 660 only.

We start with a few easy observations:

- Since $f(2) 2f(1) = -2f(1) \in \{0, 1\}$ it follows f(1) = 0.
- Since $f(3) f(1) f(2) = f(3) \in \{0, 1\}$ it follows f(3) = 1.

Now, we have the following general claim.

Claim. We must have $f(n+3) \ge f(n) + 1$. Hence, by induction, $f(3k) \ge k$ for every $k \ge 0$.

Proof.

$$0 \le f(n+3) - f(n) - f(3) = f(n+3) - (f(n)+1).$$

However, since f(9999) = 3333 exactly, it follows that equality must hold in the claim up until then. In other words, we have that f(3k) = k for each $k \ge 0$.

To finish, use the following calculation:

$$\begin{aligned} 3333 &= f(9999) \geq 3f(1982) + f(4053) = 3f(1982) + 1351 \\ \implies f(1982) \leq \frac{1982}{3} \\ \text{but also } f(1982) \geq f(1980) + f(2) = 660 + 0 = 660. \end{aligned}$$

This implies f(1982) = 660.