Twitch 036.1

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Twitch Solves ISL

Episode 36

Problem

Megumin is given 6 numbers $1, 2, 3, 4, 5, 6$ in the whiteboard. Every turn, she can choose 2 distinct numbers $x \neq y$ and replace them with the two numbers

$$\frac{xy}{|x-y|} \quad \text{and} \quad \frac{\max\{x,y\}}{2}.$$

Determine the minimum possible number that could appear in the whiteboard.

Video

https://youtu.be/8HVWommo3ZU

Solution

Rather than working with number x itself, we instead work with the number $\frac{60}{x}$. Hence we may think of operation as

- Start with 60, 30, 20, 15, 12, 10.
- Replace a, b with $|a-b|$ and $2\min\{a,b\}$.
- Try to maximize the number on board.

Claim. The sum of numbers on board is constant.

Proof. Clear. □

Claim. The numbers on the board are always positive integers.

Proof. Clear. □

Claim. There is at most one odd integer on the board at all times.

Proof. This is true at the start, and the operation preserves this (it is impossible to get more odd integers than one started with in the operation). □

The sum of the numbers on the board is initially $10+12+15+20+30+60=147$, so we cannot get a number larger than $147-(1+2+2+2+2)=138$.

This turns out to be actually achievable though: the following ad-hoc construction achieves it. In each step we repeatedly operate on two numbers to change the values.

- 10 12 15 20 30 60
- 2 20 15 20 30 60
- 4 18 15 20 30 60
- 8 18 15 16 30 60
- 16 18 15 16 22 60
- 32 18 15 16 22 44

Re-order the values for legibility and continue:

- 15 16 18 22 32 44
- 2 29 18 22 32 44
- 2 2 45 22 32 44
- 2 2 2 22 75 44
- 2 2 2 2 95 44
- 2 2 2 2 2 137
- 1 2 2 2 2 138

So the maximum 138 can indeed be reached.

Hence the answer to the original problem is $\frac{60}{138}=\frac{10}{23}$.