

# JMO 2013/4

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TWITCH SOLVES ISL

Episode 36

## Problem

Let  $f(n)$  be the number of ways to write  $n$  as a sum of powers of 2, where we keep track of the order of the summation. For example,  $f(4) = 6$  because 4 can be written as 4,  $2 + 2$ ,  $2 + 1 + 1$ ,  $1 + 2 + 1$ ,  $1 + 1 + 2$ , and  $1 + 1 + 1 + 1$ . Find the smallest  $n$  greater than 2013 for which  $f(n)$  is odd.

## Video

<https://youtu.be/Ewp4nw6Mbr4>

## External Link

<https://aops.com/community/p3043748>

## Solution

The answer is 2047.

For convenience, we agree that  $f(0) = 1$ . Then by considering cases on the first number in the representation, we derive the recurrence

$$f(n) = \sum_{k=0}^{\lfloor \log_2 n \rfloor} f(n - 2^k). \quad (\heartsuit)$$

We wish to understand the parity of  $f$ . The first few values are

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 1 \\ f(2) &= 2 \\ f(3) &= 3 \\ f(4) &= 6 \\ f(5) &= 10 \\ f(6) &= 18 \\ f(7) &= 31. \end{aligned}$$

Inspired by the data we make the key claim that

**Claim.**  $f(n)$  is odd if and only if  $n + 1$  is a power of 2.

*Proof.* We call a number *repetitive* if it is zero or its binary representation consists entirely of 1's. So we want to prove that  $f(n)$  is odd if and only if  $n$  is repetitive.

This only takes a few cases:

- If  $n = 2^k$ , then  $(\heartsuit)$  has exactly two repetitive terms on the right-hand side, namely 0 and  $2^k - 1$ .
- If  $n = 2^k + 2^\ell - 1$ , then  $(\heartsuit)$  has exactly two repetitive terms on the right-hand side, namely  $2^{\ell+1} - 1$  and  $2^\ell - 1$ .
- If  $n = 2^k - 1$ , then  $(\heartsuit)$  has exactly one repetitive terms on the right-hand side, namely  $2^{k-1} - 1$ .
- For other  $n$ , there are no repetitive terms at all on the right-hand side of  $(\heartsuit)$ .

Thus the induction checks out. □

So the final answer to the problem is 2047.