JMO 2013/4

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Twitch Solves ISL

Episode 36

Problem

Let $f(n)$ be the number of ways to write n as a sum of powers of 2, where we keep track of the order of the summation. For example, $f(4) = 6$ because 4 can be written as 4, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$. Find the smallest n greater than 2013 for which $f(n)$ is odd.

Video

https://youtu.be/Ewp4nw6Mbr4

Solution

For convenience, we agree that $f(0) = 1$. Then by considering cases on the first number in the representation, we derive the recurrence

$$f(n) = \sum_{k=1}^{\lfloor \log_2 n \rfloor} f(n - 2^k). \qquad (\heartsuit)$$

We wish to understand the parity of f. The first few values are

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 1 \\ f(2) &= 2 \\ f(3) &= 3 \\ f(4) &= 6 \\ f(5) &= 10 \\ f(6) &= 18 \\ f(7) &= 31. \end{aligned}$$

Inspired by the data we make the key claim that

Claim. $f(n)$ is odd if and only if $n + 1$ is a power of 2.

Proof. We call a number *repetitive* if it is zero or its binary representation consists entirely of 1's. So we want to prove that $f(n)$ is odd if and only if n is repetitive.

This only takes a few cases:

- If $n = 2^k$, then $(\heartsuit)$ has exactly two boring terms on the RHS, namely 0 and $2^k - 1$.
- If $n = 2^k + 2^\ell - 1$, then $(\heartsuit)$ has exactly two boring terms on the RHS, namely $2^{\ell+1} - 1$ and $2^\ell - 1$.
- If $n = 2^k - 1$, then $(\heartsuit)$ has exactly one boring terms on the RHS, namely $2^{k-1} - 1$.
- For any other of n, there are no boring terms at all on the RHS of $(\heartsuit)$.

Thus the induction checks out. □

The final answer to the problem is 2047.